Tsirelson’s problem and linear system games

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includes joint work with Richard Cleve and Li Liu
A speculative question

Conventional wisdom: Finite time / volume / energy / etc. \(\implies\) can always describe nature by finite-dimensional Hilbert spaces

But... many models in quantum mechanics and quantum field theory require infinite-dimensional Hilbert spaces (e.g. CCR)

Could nature be “intrinsically” infinite-dimensional?

Answer: Probably not

But if it was... could we recognize that fact in an experiment?

(For instance, in a Bell-type experiment?)
Non-local games (aka Bell-type experiments)

Win/lose based on outputs $a, b$ and inputs $x, y$

Alice and Bob must cooperate to win

Winning conditions known in advance

Complication: players cannot communicate while the game is in progress
Suppose game is played many times, with inputs drawn from some public distribution $\pi$

To outside observer, Alice and Bob’s strategy is described by:

$$P(a, b|x, y) = \text{the probability of output } (a, b) \text{ on input } (x, y)$$

**Correlation matrix:** collection of numbers $\{P(a, b|x, y)\}$
Non-local games ct’d

\[ P(a, b|x, y) = \text{the probability of output } (a, b) \text{ on input } (x, y) \]

With \( n \) possible questions and \( m \) possible answers, correlation matrix \( \{P(a, b|x, y)\} \) is list of \( m^2 n^2 \) probabilities

Value of game: \( \omega_C = \text{optimal winning probability using correlations } \{P(a, b|x, y)\} \text{ from fixed set of correlations } C \)

Idea: if \( \omega_{C_1} < \omega_{C_2} \) then

- there is a correlation \( \{P(a, b|x, y)\} \) in \( C_2 \) but not in \( C_1 \), and
- there is a (theoretical) Bell-type experiment to show this.
Non-local games ct’d

Value of game $\omega_C = \text{optimal winning probability using strategies}\ \{P(a, b|x, y)\} \text{ from } C$

Idea: if $\omega_{C_1} < \omega_{C_2}$ for some game then

• there is a correlation $\{P(a, b|x, y)\}$ in $C_2$ but not in $C_1$, and
• there is a (theoretical) Bell-type experiment to show this.

Bell’s theorem:

• $C_c = \text{classical correlation matrices of the form}$

$$P(a, b|x, y) = \sum \lambda_i A_i(a|x) B_i(b|y).$$

• $C_q = \text{quantum correlations}$

Then there are games with $\omega_c < \omega_q$. 

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Quantum strategies

How do we describe a quantum strategy?

Use axioms of quantum mechanics:

- Physical system described by Hilbert space
- No communication $\Rightarrow$ Alice and Bob each have their own (finite dimensional) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$
- Hilbert space for composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Shared quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$
- Alice’s output on input $x$ is modelled by measurement operators $\{M^x_a\}_a$ on $\mathcal{H}_A$
- Similarly Bob has measurement operators $\{N^y_b\}_b$ on $\mathcal{H}_B$

Quantum correlation: $P(a, b|x, y) = \langle \psi | M^x_a \otimes N^y_b | \psi \rangle$
Quantum correlations

Compare: finite and infinite dimensional correlations

\[ C_q = \left\{ \{P(a, b|x, y)\} : P(a, b|x, y) = \langle \psi | M^x_a \otimes N^y_b | \psi \rangle \right\} \]

where

\[ |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \text{ where } \mathcal{H}_A, \mathcal{H}_B \text{ fin dim'1} \]

\[ M^x_a \text{ and } N^y_b \text{ are projections on } \mathcal{H}_A \text{ and } \mathcal{H}_B \]

\[ \sum_a M^x_a = I \text{ and } \sum_b N^y_b = I \text{ for all } x, y \]

and

\[ C_{qs} = \text{same but } \mathcal{H}_A \text{ and } \mathcal{H}_B \text{ are possibly infinite-dimensional} \]

Question: is \( \omega_q < \omega_{qs} \) for some game?

Answer: No!
Quantum correlations ct’d

$C_q$ : finite-dimensional correlations

$C_{qs}$: possibly-infinite-dimensional correlations

Question: is $\omega_q < \omega_{qs}$ for some game?

Answer: No!

Let $C_{qa} = \overline{C_q}$, so $\omega_q = \omega_{qa}$ for any game.

Then $C_q \subseteq C_{qs} \subseteq C_{qa}$, so $\omega_{qs} = \omega_{qa} = \omega_q$ for any game.

Fortunately, this is not the end of the story

We’ve assumed that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$... maybe this is too restrictive
Commuting-operator model

Another model for composite systems: \( \textit{commuting-operator model} \)

In this model:

- Alice and Bob each have an algebra of observables \( \mathcal{A} \) and \( \mathcal{B} \)
- \( \mathcal{A} \) and \( \mathcal{B} \) act on the joint Hilbert space \( \mathcal{H} \)
- \( \mathcal{A} \) and \( \mathcal{B} \) commute: if \( a \in \mathcal{A} \), \( b \in \mathcal{B} \), then \( ab = ba \).

This model is used (for instance) in quantum field theory.

Correlation set:

\[
C_{qc} := \left\{ \{P(a, b|x, y)\} : P(a, b|x, y) = \langle \psi | M_a^x N_b^y | \psi \rangle, \quad M_a^x N_b^y = N_b^y M_a^x \right\}
\]

Hierarchy: \( C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc} \)
Tsirelson’s problem

Two models of QM: tensor product and commuting-operator

Tsirelson problems: is $C_t$, $t \in \{q, qs, qa\}$ equal to $C_{qc}$

Fundamental questions:

1. What is the power of these models?
   
   Strong Tsirelson: is $C_q = C_{qc}$?

2. Is $\omega_{qa} < \omega_{qc}$ for any game?
   
   Since $C_{qa}$ and $C_{qc}$ are closed convex sets, equivalent to
   
   Weak Tsirelson: is $C_{qa} = C_{qc}$?
What do we know?

\[ C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc} \]

Strong

weak

Theorem (Ozawa, JNPPSW, Fr)

\[ C_{qa} = C_{qc} \text{ if and only if Connes' embedding problem is true} \]

Theorem (S)

\[ C_{qs} \neq C_{qc} \]
Other fundamental questions

1. Given a non-local game, can we compute the optimal value $\omega_t$ over strategies in $C_t$, $t \in \{qa, qc\}$?

2. Is $C_q = C_{qa}$? In other words:
   - Is $C_q$ closed?
   - Does every non-local game have an optimal finite-dimensional strategy?

3. Given $P \in C_q$, is there a computable upper bound on the dimension needed to realize $P$?
What do we know?

Question: Given a non-local game, can we compute the optimal value $\omega_t$ over strategies in $C_t$, $t \in \{qa, qc\}$?

Brute force: search through strategies on $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^n$, converges to $\omega_{qa}$ (from below)

Navascués, Pironio, Acín: Given a non-local game, there is a hierarchy of SDPs which converge in value to $\omega_{qc}$ (from above)

Problem: in both cases, no way to tell how close we are to the correct answer

Theorem (S)

*It is undecidable to tell if $\omega_{qc} < 1$*

General cases of other questions completely open!
Undecidability of quantum logic

**Theorem (S)**

*It is undecidable to tell if $\omega_{qc} < 1*

We’re now talking about something practical (but mathematics, not physics)

More generally: Can we tell if a statement like

Conditions on collection of projections $\implies$ some other condition on the collection

holds in all Hilbert spaces.

T. Fritz: Proof of above theorem implies quantum logic is undecidable
Two theorems

Theorem (S)

\[ C_{qs} \neq C_{qc} \]

Theorem (S)

*It is undecidable to tell if* \( \omega_{qc} < 1 \)

Theorems look very different...

But: proof follows from a single theorem in group theory

Connection with group theory comes from linear system games
Linear system games

Start with $m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$

Inputs:
- Alice receives $1 \leq i \leq m$ (an equation)
- Bob receives $1 \leq j \leq n$ (a variable)

Outputs:
- Alice outputs an assignment $a_k$ for all variables $x_k$ with $A_{ik} \neq 0$
- Bob outputs an assignment $b_j$ for $x_j$

They win if:
- $A_{ij} = 0$ (assignment irrelevant) or
- $A_{ij} \neq 0$ and $a_j = b_j$ (assignment consistent)
Quantum solutions of $Ax = b$

Observables $X_j$ such that

1. $X_j^2 = I$ for all $j$
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for all $i$
3. If $A_{ij}, A_{ik} \neq 0$, then $X_jX_k = X_kX_j$

(We’ve written linear equations multiplicatively)

Theorem (Cleve-Mittal, Cleve-Liu-S)

Let $G$ be the game for linear system $Ax = b$. Then:

- $G$ has a perfect strategy in $C_{qs}$ if and only if $Ax = b$ has a finite-dimensional quantum solution
- $G$ has a perfect strategy in $C_{qc}$ if and only if $Ax = b$ has a quantum solution
Group theory

Finitely presented group: algebraic structure generated by products of symbols

\[ X_1, \ldots, X_n \]

and their inverses, satisfying some relations

\[ R_i(X_1, \ldots, X_n), 1 \leq i \leq k \]

Important: the empty product \( e \) acts like 1, but no scalars (i.e. no \(-1\)), no addition

Example:

\[ \langle a, b : a^2 = b^2 = e = (ab)^3 \rangle \]

Question: which relations determine interesting groups?
Group theory ct’d

Is there a group generated by $X_1, \ldots, X_n$ satisfying the quantum linear relations from before?

1. $X_j^2 = I$ for all $j$
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for all $i$
3. If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

Almost: replace $-I$ with new symbol $J$

The solution group $\Gamma$ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

1. $X_j^2 = [X_j, J] = J^2 = e$ for all $j$
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = J^{b_i}$ for all $i$
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$

where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity
Group theory ct’d

The solution group $\Gamma$ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

1. $X_j^2 = [X_j, J] = J^2 = e$ for all $j$
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where $[a, b] = aba^{-1}b^{-1}$, $e = \text{group identity}$

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let $G$ be the game for linear system $Ax = b$. Then:

- $G$ has a perfect strategy in $C_{qs}$ if and only if $\Gamma$ has a finite-dimensional representation with $J \neq I$
- $G$ has a perfect strategy in $C_{qc}$ if and only if $J \neq e$ in $\Gamma$
Groups and local compatibility

Suppose we can write down any group relations we want...

But: generators in the relation will be forced to commute!

Call this condition *local compatibility*

Local compatibility is (a priori) a very strong constraint

For instance, $S_3$ is generated by $a, b$ subject to the relations

$$a^2 = b^2 = e, (ab)^3 = e$$

If $ab = ba$, then $(ab)^3 = a^3 b^3 = ab$

So relations imply $a = b$, and $S_3$ becomes $\mathbb{Z}_2$
Group embedding theorem

Solution groups satisfy local compatibility

Nonetheless:

Solution groups are as complicated as general groups

**Theorem (S)**

Let $G$ be any finitely-presented group, and suppose we are given $J_0$ in the center of $G$ such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \rightarrow \Gamma$, where $\Gamma$ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$. 
How do we prove the embedding theorem?

Linear system $Ax = b$ over $\mathbb{Z}_2$ equivalent to labelled hypergraph:

Edges are variables

Vertices are equations

$v$ is adjacent to $e$ if and only if $A_{ve} \neq 0$

$v$ is labelled by $b_i \in \mathbb{Z}_2$

Given finitely-presented group $G$, we get $\Gamma$ from a linear system

But what linear system?

Can answer this pictorially by writing down a hypergraph?
The hypergraph by example

\[ \langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle \]
\[
\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle
\]

Thank-you!