Tsirelson's problem and linear system games

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includes joint work with Richard Cleve and Li Liu

A speculative question

Conventional wisdom: Finite time / volume / energy / etc. $\Longrightarrow$ can always describe nature by finite-dimensional Hilbert spaces

But... many models in quantum mechanics and quantum field theory require infinite-dimensional Hilbert spaces (e.g. CCR)

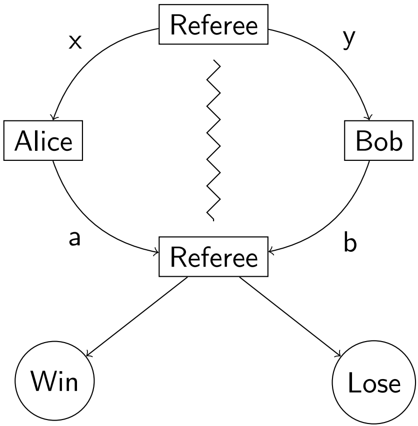
Could nature be "intrinsically" infinite-dimensional?

Answer: Probably not

But if it was... could we recognize that fact in an experiment?

(For instance, in a Bell-type experiment?)

Non-local games (aka Bell-type experiments)



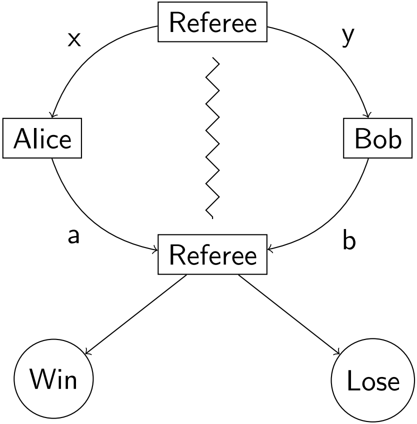
Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Complication: players cannot communicate while the game is in progress

Non-local games ct'd



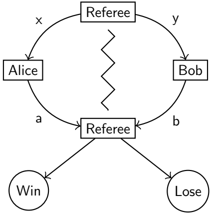
Suppose game is played many times, with inputs drawn from some public distribution π

To outside observer, Alice and Bob's strategy is described by:

$P(a, b|x, y) =$ the probability of output (a, b) on input (x, y)

Correlation matrix: collection of numbers $\{P(a, b|x, y)\}$

Non-local games ct'd



$P(a, b|x, y) =$ the probability of output (a, b) on input (x, y)

With n possible questions and m possible answers, correlation matrix $\{P(a, b|x, y)\}$ is list of $m^2 n^2$ probabilities

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Value of game: $\omega_C =$ optimal winning probability using correlations $\{P(a, b|x, y)\}$ from fixed set of correlations C

Idea: if $\omega_{C_1} < \omega_{C_2}$ then

- there is a correlation $\{P(a, b|x, y)\}$ in C_2 but not in C_1, and
- there is a (theoretical) Bell-type experiment to show this.

Non-local games ct'd

Value of game ω_C = optimal winning probability using strategies $\{P(a,b|x,y)\}$ from C

Idea: if $\omega_{C_1} < \omega_{C_2}$ for some game then

- there is a correlation $\{P(a,b|x,y)\}$ in C_2 but not in C_1, and
- there is a (theoretical) Bell-type experiment to show this.

Bell's theorem:

- C_c = classical correlation matrices of the form

$$P(a,b|x,y) = \sum \lambda_i A_i(a|x) B_i(b|y).$$

- C_q = quantum correlations

Then there are games with $\omega_c < \omega_q$.

Quantum strategies

How do we describe a quantum strategy?

Use axioms of quantum mechanics:

- Physical system described by Hilbert space
- No communication $\Rightarrow$ Alice and Bob each have their own (finite dimensional) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$
- Hilbert space for composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Shared quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$
- Alice's output on input x is modelled by measurement operators $\{M_a^x\}_a$ on $\mathcal{H}_A$
- Similarly Bob has measurement operators $\{N_b^y\}_b$ on $\mathcal{H}_B$

Quantum correlation: $P(a, b|x, y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle$

Quantum correlations

Compare: finite and infinite dimensional correlations

$$C_q = \Big\{\{P(a,b|x,y)\} : P(a,b|x,y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle \text{ where}$$
$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \text{ where } \mathcal{H}_A, \mathcal{H}_B \text{ fin dim'l}$$
$$M_a^x \text{ and } N_b^y \text{ are projections on } \mathcal{H}_A \text{ and } \mathcal{H}_B$$
$$\sum_a M_a^x = I \text{ and } \sum_b N_b^y = I \text{ for all } x, y\Big\}$$

and

$C_{qs} =$ same but $\mathcal{H}_A$ and $\mathcal{H}_B$ are possibly infinite-dimensional

Question: is $\omega_q < \omega_{qs}$ for some game?

Answer: No!

Quantum correlations ct'd

C_q : finite-dimensional correlations

C_{qs}: possibly-infinite-dimensional correlations

Question: is $\omega_q < \omega_{qs}$ for some game?

Answer: No!

Let $C_{qa} = \overline{C_q}$, so $\omega_q = \omega_{qa}$ for any game

Then $C_q \subseteq C_{qs} \subseteq C_{qa}$, so $\omega_{qs} = \omega_{qa} = \omega_q$ for any game.

Fortunately, this is not the end of the story

We've assumed that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$... maybe this is too restrictive

Commuting-operator model

Another model for composite systems: *commuting-operator model*

In this model:

- Alice and Bob each have an algebra of observables $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A}$ and $\mathcal{B}$ act on the joint Hilbert space $\mathcal{H}$
- $\mathcal{A}$ and $\mathcal{B}$ commute: if $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $ab = ba$.

This model is used (for instance) in quantum field theory

Correlation set:

$$C_{qc} := \Big\{ \{P(a,b|x,y)\} : P(a,b|x,y) = \langle\psi| M_a^x N_b^y |\psi\rangle, \quad M_a^x N_b^y = N_b^y M_a^x \Big\}$$

Hierarchy: $C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$

Tsirelson's problem

strong

$$C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$$

weak

Two models of QM: tensor product and commuting-operator

Tsirelson problems: is C_t, $t \in \{q, qs, qa\}$ equal to C_{qc}

Fundamental questions:

1. What is the power of these models?

   Strong Tsirelson: is $C_q = C_{qc}$?

2. Is $\omega_{qa} < \omega_{qc}$ for any game?

   Since C_{qa} and C_{qc} are closed convex sets, equivalent to

   Weak Tsirelson: is $C_{qa} = C_{qc}$?

What do we know?

$$\overbrace{C_q \subseteq C_{qs} \subseteq \underbrace{C_{qa} \subseteq C_{qc}}_{\text{weak}}}^{\text{strong}}$$

Theorem (Ozawa, JNPPSW, Fr)

$C_{qa} = C_{qc}$ if and only if Connes' embedding problem is true

Theorem (S)

$C_{qs} \neq C_{qc}$

Other fundamental questions

1. Given a non-local game, can we compute the optimal value ω_t over strategies in C_t, $t \in \{qa, qc\}$?

2. Is $C_q = C_{qa}$? In other words:
   - Is C_q closed?
   - Does every non-local game have an optimal finite-dimensional strategy?

3. Given $P \in C_q$, is there a computable upper bound on the dimension needed to realize P?

What do we know?

Question: Given a non-local game, can we compute the optimal value ω_t over strategies in C_t, $t \in \{qa, qc\}$?

Brute force: search through strategies on $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^n$, converges to ω_{qa} (from below)

Navascués, Pironio, Acín: Given a non-local game, there is a hierarchy of SDPs which converge in value to ω_{qc} (from above)

Problem: in both cases, no way to tell how close we are to the correct answer

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

General cases of other questions completely open!

Undecidability of quantum logic

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

We're now talking about something practical (but mathematics, not physics)

More generally: Can we tell if a statement like

$$\text{Conditions on collection of projections} \implies \text{some other condition on the collection}$$

holds in all Hilbert spaces.

T. Fritz: Proof of above theorem implies quantum logic is undecidable

Two theorems

Theorem (S)

$C_{qs} \neq C_{qc}$

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

Theorems look very different...

But: proof follows from a single theorem in group theory

Connection with group theory comes from linear system games

Linear system games

Start with $m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$

Inputs:

- Alice receives $1 \leq i \leq m$ (an equation)
- Bob receives $1 \leq j \leq n$ (a variable)

Outputs:

- Alice outputs an assignment a_k for all variables x_k with $A_{ik} \neq 0$
- Bob outputs an assignment b_j for x_j

They win if:

- $A_{ij} = 0$ (assignment irrelevant) or
- $A_{ij} \neq 0$ and $a_j = b_j$ (assignment consistent)

Quantum solutions of $Ax = b$

Observables X_j such that

1. $X_j^2 = I$ for all j
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

(We've written linear equations multiplicatively)

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if $Ax = b$ has a finite-dimensional quantum solution*
- *G has a perfect strategy in C_{qc} if and only if $Ax = b$ has a quantum solution*

Group theory

Finitely presented group: algebraic structure generated by products of symbols

$$X_1, \ldots, X_n$$

and their inverses, satisfying some relations

$$R_i(X_1, \ldots, X_n), 1 \leq i \leq k$$

Important: the empty product e acts like 1, but

no scalars (i.e. no -1), no addition

Example:

$$\langle a, b : a^2 = b^2 = e = (ab)^3 \rangle$$

Question: which relations determine interesting groups?

Group theory ct'd

Is there a group generated by $X_1, \ldots, X_n$ satisfying the quantum linear relations from before?

1. $X_j^2 = I$ for all j
2. $\prod_{j=1}^n X_j^{A_{ij}} = (-I)^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

Almost: replace $-I$ with new symbol J

The *solution group* Γ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

1. $X_j^2 = [X_j, J] = J^2 = e$ for all j
2. $\prod_{j=1}^n X_j^{A_{ij}} = J^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$

where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity

Group theory ct'd

The *solution group* Γ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

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where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- *G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Groups and local compatibility

Suppose we can write down any group relations we want...

But: generators in the relation will be forced to commute!

Call this condition *local compatibility*

Local compatibility is (a priori) a very strong constraint

For instance, S_3 is generated by a, b subject to the relations

$$a^2 = b^2 = e, (ab)^3 = e$$

If $ab = ba$, then $(ab)^3 = a^3 b^3 = ab$

So relations imply $a = b$, and S_3 becomes $\mathbb{Z}_2$

Group embedding theorem

Solution groups satisfy local compatibility

Nonetheless:

Solution groups are as complicated as general groups

Theorem (S)

Let G be any finitely-presented group, and suppose we are given J_0 in the center of G such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where Γ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$.

How do we prove the embedding theorem?

Linear system $Ax = b$ over $\mathbb{Z}_2$ equivalent to labelled hypergraph:

Edges are variables

Vertices are equations

v is adjacent to e if and only if $A_{ve} \neq 0$

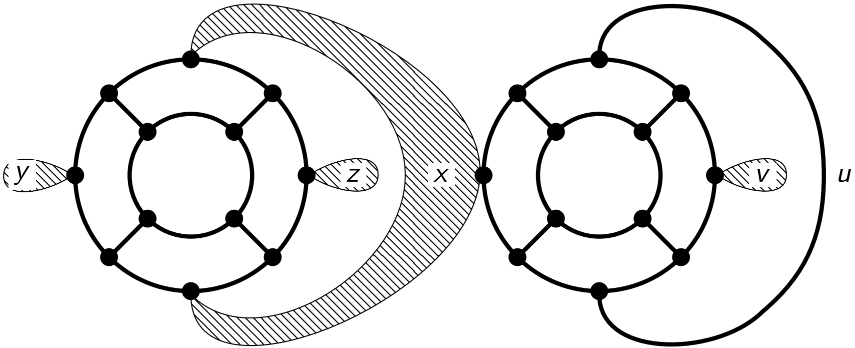
v is labelled by $b_i \in \mathbb{Z}_2$

Given finitely-presented group G, we get Γ from a linear system

But what linear system?

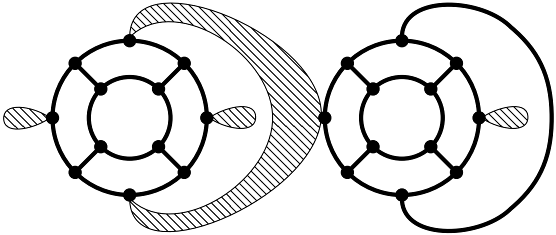
Can answer this pictorially by writing down a hypergraph?

The hypergraph by example



$$\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle$$

The end



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Thank-you!