Tsirelson's problem and linear system games

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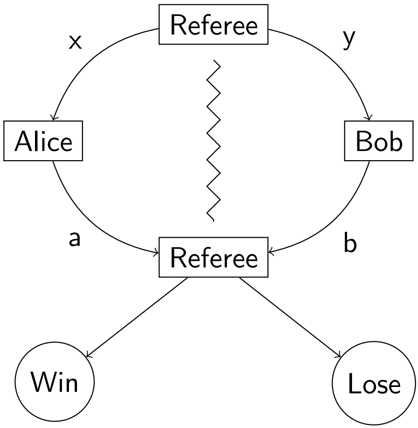
(with some corrections)

includes joint work with Richard Cleve and Li Liu

Non-local games



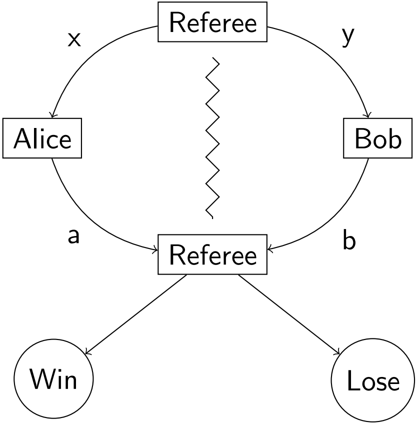
Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Complication: players cannot communicate while the game is in progress

Strategies for non-local games



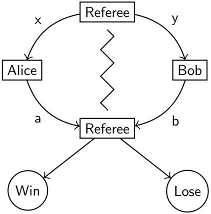
Suppose game is played many times, with inputs drawn from some public distribution π

To outside observer, Alice and Bob's strategy is described by:

$P(a, b|x, y)$ = the probability of output (a, b) on input (x, y)

Correlation matrix: collection of numbers $\{P(a, b|x, y)\}$

Classical and quantum strategies



$P(a, b|x, y) =$ the probability of output (a, b) on input (x, y)

Value of game $\omega =$ winning probability using strategy $\{P(a, b|x, y)\}$

What type of strategies might Alice and Bob use?

Classical: can use randomness, flip coin to determine output.

Correlation matrix will be $P(a, b|x, y) = A(a|x)B(b|y)$.

Quantum: Alice and Bob can share entangled quantum state

Bell's theorem: Alice and Bob can do better with an entangled quantum state than they can do classically

Quantum strategies

How do we describe a quantum strategy?

Use axioms of quantum mechanics:

- Physical system described by (finite-dimensional) Hilbert space
- No communication $\Rightarrow$ Alice and Bob each have their own (finite dimensional) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$
- Hilbert space for composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Shared quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$
- Alice's output on input x is modelled by measurement operators $\{M_a^x\}_a$ on $\mathcal{H}_A$
- Similarly Bob has measurement operators $\{N_b^y\}_b$ on $\mathcal{H}_B$

Quantum correlation: $P(a, b|x, y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle$

Quantum correlations

Set of quantum correlations:

$$C_q = \Big\{ \{P(a,b|x,y)\} : P(a,b|x,y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle \text{ where}$$
$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \text{ where } \mathcal{H}_A, \mathcal{H}_B \text{ fin dim'l}$$
$$M_a^x \text{ and } N_b^y \text{ are projections on } \mathcal{H}_A \text{ and } \mathcal{H}_B$$
$$\sum_a M_a^x = I \text{ and } \sum_b N_b^y = I \text{ for all } x, y \Big\}$$

Two variants:

1. C_{qs}: Allow $\mathcal{H}_A$ and $\mathcal{H}_B$ to be infinite-dimensional
2. $C_{qa} = \overline{C_q}$: limits of finite-dimensional strategies

Relations: $C_q \subseteq C_{qs} \subseteq C_{qa}$

Commuting-operator model

Another model for composite systems: *commuting-operator model*

In this model:

- Alice and Bob each have an algebra of observables $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A}$ and $\mathcal{B}$ act on the joint Hilbert space $\mathcal{H}$
- $\mathcal{A}$ and $\mathcal{B}$ commute: if $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $ab = ba$.

This model is used in quantum field theory

Correlation set:

$$C_{qc} := \Big\{ \{P(a,b|x,y)\} : P(a,b|x,y) = \langle \psi | M_a^x N_b^y | \psi \rangle , \\ M_a^x N_b^y = N_b^y M_a^x \Big\}$$

Hierarchy: $C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$

Tsirelson's problem

$$\begin{tikzcd} C_q \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend left, dash, "\text{strong}"] & C_{qs} \arrow[r, phantom, "\subseteq"] & C_{qa} \arrow[r, phantom, "\subseteq"] \arrow[r, bend right, dash, "\text{weak}"'] & C_{qc} \end{tikzcd}$$

Two models of QM: tensor product and commuting-operator

Tsirelson problems: is C_t, $t \in \{q, qs, qa\}$ equal to C_{qc}

Fundamental questions:

1. What is the power of these models?

   Strong Tsirelson: is $C_q = C_{qc}$?

2. Are there observable differences between these two models, accounting for noise and experimental error?

   Weak Tsirelson: is $C_{qa} = C_{qc}$?

What do we know?

$$\begin{tikzcd} C_q \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend left, dash, "\text{strong}"] & C_{qs} \arrow[r, phantom, "\subseteq"] & C_{qa} \arrow[r, phantom, "\subseteq"] \arrow[r, bend right, dash, "\text{weak}"'] & C_{qc} \end{tikzcd}$$

Theorem (Ozawa, JNPPSW, Fr)

$C_{qa} = C_{qc}$ *if and only if Connes' embedding problem is true*

Theorem (S)

$C_{qs} \neq C_{qc}$

Other formulations (NCPV)

Formulation due to Navascués, Cooney, Pérez-García, Villanueva

Given $\{P(a,b|x,y)\}$

Local measurement statistics: $P(a|x) = \sum_{b,y} P(a,b|x,y)$,
$P(b|y) = \text{similar}$

Rather than modeling joint system, model Bob's system:

1. For local measurement statistics, find measurements $\{N_b^y\}$ and density matrix ρ such that

$$P(b|y) = \operatorname{tr}\left(N_b^y \rho\right)$$

2. For joint statistics, find measurements $\{N_b^y\}$ and density matrices ρ^{xa} such that

$$P(a,b|x,y) = P(a|x) \operatorname{tr}\left(N_b^y \rho^{xa}\right).$$

Other formulations (NCPV continued)

1. For local measurement statistics, find measurements $\{N_b^y\}$ and density matrix ρ such that

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2. For joint statistics, find measurements $\{N_b^y\}$ and density matrices ρ^{xa} such that

$$P(a, b|x, y) = P(a|x) \operatorname{tr}\left(N_b^y \rho^{xa}\right).$$

Question: Can Bob build a model of his local statistics which is consistent with Alice's observed inputs/outputs?

Answer: If and only if there are ρ^{xa} as above with $\sum_a P(a|x)\rho^{xa} = \rho$ (independent of x)

Other formulations (NCPV continued)

Question: Can Bob build a model of his local statistics which is consistent with Alice's observed inputs/outputs?

Answer: If and only if there are ρ^{xa} as above with $\sum_a P(a|x)\rho^{xa} = \rho$ (independent of x)

Fact: This happens if and only if $\{P(a,b|x,y)\}$ belongs to C_{qs}

General state: a linear functional $f : \mathcal{B} \to \mathbb{C}$ such that $f(I) = 1$
and $f(A) \geq 0$ if A is positive

If ρ density matrix, then $f(A) = \mathrm{tr}(A\rho)$ is general state

Not every general state comes from a density matrix

What if Bob uses general states instead of density matrices?

Other formulations (NCPV continued)

Condition (*): Bob can build a model of his local statistics which is consistent with Alice's observed inputs/outputs

If Bob uses density matrices, then (*) holds if and only if $\{P(a,b|x,y)\}$ belongs to C_{qs}

If Bob uses general states, then (*) holds if and only if $\{P(a,b|x,y)\}$ belongs to C_{qc}

Conclusion:

Since $C_{qs} \neq C_{qc}$, modeling power of general states is greater than modeling power of density matrices, even for Bell scenarios

Other formulations (Ozawa)

Correlations with limited interactions:

$$C_{qc}(\epsilon) = \Big\{ \{P(a,b|x,y)\} : P(a,b|x,y) = \langle\psi| M_a^x \circ N_b^y |\psi\rangle \\ \left\| M_a^x N_b^y - N_b^y M_a^x \right\| \leq \epsilon \\ |\psi\rangle \in \text{finite-diml } \mathcal{H} \Big\}$$

These correlations are non-signalling

Theorem (Ozawa,Coudron-Vidick)

$C_{qc} = \bigcap_{\epsilon > 0} C_{qc}(\epsilon)$

If $\{P(a,b|x,y)\}$ has finite-dimensional limited interaction models for every $\epsilon > 0$, does it belong to C_q or C_{qa}? (Answer: no)

Other fundamental questions

1. Given a non-local game, can we compute the optimal value ω_t over strategies in C_t, $t \in \{qa, qc\}$?

2. Is $C_q = C_{qa}$? (In other words, does every non-local game have an optimal finite-dimensional strategy?)

3. Given $P \in C_q$, is there a computable upper bound on the dimension needed to realize P?

What do we know?

Theorem (Navascués, Pironio, Acín)

Given a non-local game, there is a hierarchy of SDPs which converge in value to ω_{qc}

Problem: no way to tell how close we are to the correct answer

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

General cases of other questions completely open!

Two theorems

Theorem (S)

$C_{qs} \neq C_{qc}$

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

Proofs: make connection to group theory via linear system games

Linear system games

Start with $m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$

Inputs:

- Alice receives $1 \leq i \leq m$ (an equation)
- Bob receives $1 \leq j \leq n$ (a variable)

Outputs:

- Alice outputs an assignment a_k for all variables x_k with $A_{ik} \neq 0$
- Bob outputs an assignment b_j for x_j

They win if:

- $A_{ij} = 0$ (assignment irrelevant) or
- $A_{ij} \neq 0$ and $a_j = b_j$ (assignment consistent)

Quantum solutions of $Ax = b$

Observables X_j such that

1. $X_j^2 = I$ for all j
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

(We've written linear equations multiplicatively)

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if $Ax = b$ has a finite-dimensional quantum solution*
- *G has a perfect strategy in C_{qc} if and only if $Ax = b$ has a quantum solution*

Quantum solutions ct'd

The *solution group* Γ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

1. $X_j^2 = [X_j, J] = J^2 = e$ for all j
2. $\prod_{j=1}^n X_j^{A_{ij}} = J^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$

where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- *G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Groups and local compatibility

Suppose we can write down any group relations we want...

But: generators in the relation will be forced to commute!

Call this condition *local compatibility*

Local compatibility is (a priori) a very strong constraint

For instance, S_3 is generated by a, b subject to the relations

$$a^2 = b^2 = e, (ab)^3 = e$$

If $ab = ba$, then $(ab)^3 = a^3 b^3 = ab$

So relations imply $a = b$, and S_3 becomes $\mathbb{Z}_2$

Group embedding theorem

Solution groups satisfy local compatibility

Nonetheless:

Theorem (S)

Let G be any finitely-presented group, and suppose we are given J_0 in the center of G such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where Γ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$.

Furthermore, if $X_1, \ldots, X_n$ are some elements of G with $X_i^2 = e$, then we can also require that $\phi(X_i)$ is a generator of Γ.

Non-residually finite groups

Embedding theorem useful because there are groups with interesting properties

For instance, there are finitely-presented non-residually-finite groups:

K with an element $g \neq e$ such that $g \mapsto I$ in every finite-dimensional representation

For example, Higman's group:

$$\begin{aligned} K = \langle a, b, c, d : & aba^{-1} = b^2, bcb^{-1} = c^2, \\ & cdc^{-1} = d^2, dad^{-1} = a^2 \rangle \end{aligned}$$

Only finite-dimensional representation is the trivial representation!

Strong Tsirelson is false

Start with group K with an element $g \neq e$ such that $g \mapsto I$ in every finite-dimensional representation

Add two generators x and J_0

Add relations $[g, x] = J_0$ and $[J_0, G] = J_0^2 = 1$.

Conclusion: get a group G with a central element $J_0 \neq e$, $J_0^2 = e$, such that $J_0 \mapsto I$ in every finite-dimensional representation

Embedding theorem: embed G in a solution group Γ

$$\begin{aligned} G &\hookrightarrow \Gamma \rightarrow \mathcal{U}(n) \\ J_0 &\mapsto J \mapsto I \end{aligned}$$

Get a solution group Γ where $J \neq e$, but $J \mapsto I$ in every finite-dimensional representation

Strong Tsirelson is false (continued)

Get a solution group Γ where $J \neq e$, but $J \mapsto I$ in every finite-dimensional representation

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- *G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Game associated to Γ has a perfect strategy in C_{qc}

Does not have a perfect strategy in C_{qs}

Conclusion: $C_{qs} \neq C_{qc}$

How do we prove the embedding theorem?

Linear system $Ax = b$ over $\mathbb{Z}_2$ equivalent to labelled hypergraph:

Edges are variables

Vertices are equations

v is adjacent to e if and only if $A_{ve} \neq 0$

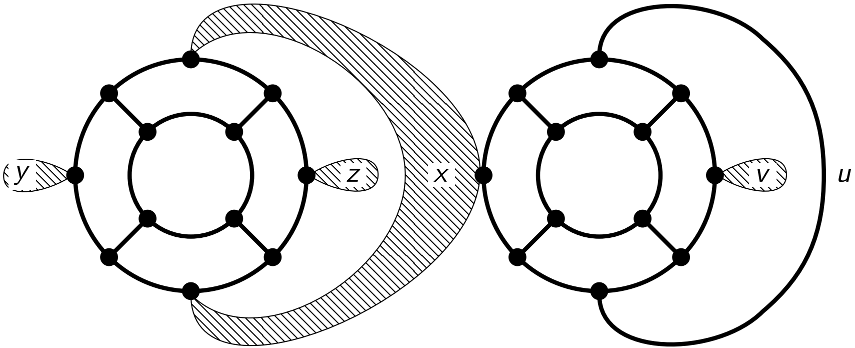
v is labelled by $b_i \in \mathbb{Z}_2$

Given finitely-presented group G, we get Γ from a linear system

But what linear system?

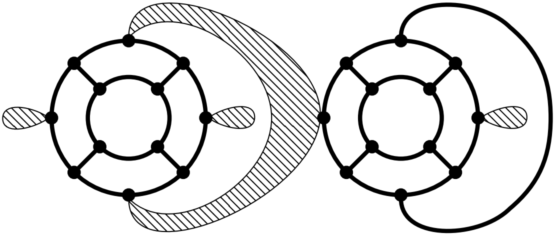
Can answer this pictorially by writing down a hypergraph?

The hypergraph by example



$$\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle$$

The end



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Thank-you!