Local formality of inversion hyperplane arrangements

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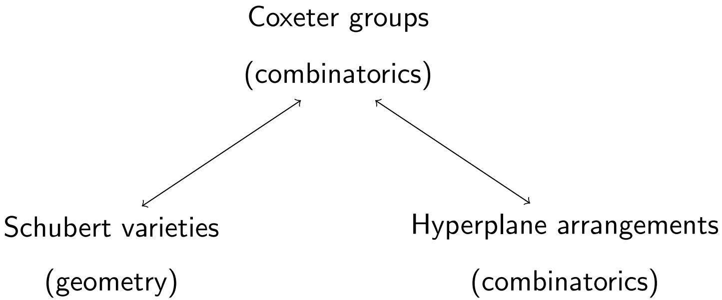
joint work with Travis Scrimshaw

Basic ideas


Coxeter groups
(combinatorics)
Schubert varieties
(geometry)
Hyperplane arrangements
(combinatorics)

Inversion hyperplane arrangements

Hyperplane arrangement: collection of subspaces of $\mathbb{R}^n$.

W : finite Weyl group, such as S_n

Build hyperplane arrangement from **inversions** of $w \in W$.

$R = R^+ \cup R^-$: root system of W.

Inversions of w: $\alpha \in R^+$ such that $w^{-1}\alpha \in R^-$.

Inversion hyperplane arrangement:

$$\mathcal{I}(w) = \bigcup_{\text{inversions } \alpha} \ker \alpha.$$

Inversion hyperplane arrangements continued

Inversions of w: $\alpha \in R^+$ such that $w^{-1}\alpha \in R^-$.

Inversion hyperplane arrangement:

$$\mathcal{I}(w) = \bigcup_{\text{inversions } \alpha} \ker \alpha.$$

Example ($W = S_4$)

$R^+ = \{e_i - e_j : 1 \leq i < j \leq 4\}$

$e_i - e_j$, $i < j$ is an inversion if and only if $w(i) > w(j)$

$\ker(e_i - e_j) = \{x \in \mathbb{C}^4 : x_i = x_j\}$.

$\mathcal{I}(3412) = (x_2 = x_3) \cup (x_1 = x_3) \cup (x_2 = x_4) \cup (x_1 = x_4) \subset \mathbb{C}^4$

Schubert varieties versus hyperplane arrangements

A hyperplane arrangement is **free** if its module of derivations is free.

Theorem (S 2015)

Schubert variety $X(w)$ is rationally smooth if and only if

$\mathcal{I}(w)$ is free and $\#$ of chambers $=$ size of Bruhat interval

How to see the connection?

rootsystem pattern avoidance

Root system pattern avoidance

Rootsystem R lives in vector space V

Root subsystem: $R_0 = R \cap V_0$ for $V_0 \subseteq V$

Weyl group $W(R_0)$ is a subgroup of W

$I(w) \cap V_0$ is the inversion set of an element $w_0 \in W(R_0)$, so

get a flattening map $\mathrm{fl}_{V_0} : W \to W(R_0)$

A **pattern** is a pair (w_1, R_1) with $w_1 \in W(R_1)$

Defn (Billey-Postnikov): $w \in W(R)$ **contains** (w_1, R_1) if

- R_1 is isomorphic to a subsystem of R
- $\mathrm{fl}_{R_0}(w) = w_1$

Generalizes permutation pattern avoidance

Root system pattern avoidance continued

Defn (Billey-Postnikov): $w \in W(R)$ contains (w_1, R_1) if

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Generalizes permutation pattern avoidance

Theorem (Lakshmibai-Sandya)

For $W = S_n$, $X(w)$ is smooth if and only if w avoids 3412 and 4231

Theorem (Billey, Billey-Postnikov)

$X(w)$ is (rationally) smooth if and only if w avoids a finite list of root system patterns.

All patterns belong to **stellar root systems**... only need three patterns to cover A, D, and E

Connection with hyperplane arrangements

Given: $\mathcal{A} = \bigcup_{\alpha \in S} \ker \alpha$

A **flat** of $\mathcal{A}$ is an intersection X of hyperplanes of $\mathcal{A}$

Flats of $\mathcal{A}$ correspond to linearly-closed subsets $S_0 = S \cap V_0$ of S

Localization of $\mathcal{A}$ is the arrangement

$$\mathcal{A}_X = \bigcup_{X \subseteq H \in \mathcal{A}} H = \bigcup_{\alpha \in S_0} \ker \alpha$$

$\mathcal{A}_X$ has rank $\operatorname{codim} X = \dim \operatorname{span} S_0$.

If $\mathcal{A} = \mathcal{I}(w)$, then $\mathcal{I}(w)_X = \mathcal{I}(\mathrm{fl}_{V_0}(w))$

Pattern avoidance criteria = check on localizations

Pattern avoidance for freeness

Theorem (S 2015)

Schubert variety $X(w)$ is rationally smooth if and only if

$\mathcal{I}(w)$ is free and $\#$ of chambers $=$ size of Bruhat interval

Theorem (S 2016)

An inversion arrangement $\mathcal{I}(w)$ is free if and only if w avoids a finite list of root system patterns.

Proof uses concept from geometry of Schubert varieties
Peterson translation

Freeness is a local property: $\mathcal{A}$ free implies $\mathcal{A}_X$ free

Question: can we do this for other local properties of $\mathcal{I}(w)$?

Terao's conjecture

Theorem (S 2016)

An inversion arrangement $\mathcal{I}(w)$ is free if and only if w avoids a finite list of root system patterns.

Conjecture (Terao)

If matroid$(\mathcal{A}) \cong$ matroid$(\mathcal{B})$ and $\mathcal{A}$ is free then $\mathcal{B}$ is free.

Weak versions of conjecture: check for $\mathcal{A}$ and $\mathcal{B}$ in some family of arrangements

Scrimshaw-S.: weak Terao's conjecture true for inversion hyperplane arrangements.

Terao's conjecture and formality

Terao's conjecture has lead to the study of many other properties (combinatorial or not) which are "close" to freeness

An arrangement is k-**generated** if cycle space spanned by k-cycles

Formal if 3-generated

Freeness implies local formality: $\mathcal{A}_X$ formal for all flats X

k-generatedness of inversion sets

Theorem (Scrimshaw-S)

Let $\sigma(R) = \min\{k :$ *for all* $w \in W$ *and* $s \in$ *span Inv*(w) *there exists* $X \subset I(w)$ *with* $s \in$ *span* X *and* $|X| \leq k\}$

Then $\min\{k : \mathcal{I}(w)$ *is* k*-generated for all* $w \in W\} = \sigma(R) + 1$ *and*

$$\sigma(A_n) = \sigma(B_n) = \sigma(C_n) = \sigma(F_4) = 3,$$
$$\sigma(D_n) = \sigma(E_n) = 4$$

Corollary (Scrimshaw-S)

$\mathcal{I}(w)$ is locally formal if and only if w avoids a finite list of root system patterns. Consequently, local formality is a combinatorial property of inversion arrangements.

k-generatedness of inversion sets ct'd

Theorem (Scrimshaw-S)

Let $\sigma(R) = \min\{k : \text{for all } w \in W \text{ and } s \in \text{span } Inv(w)$
$\text{there exists } X \subset I(w) \text{ with } s \in \text{span } X$
$\text{and } |X| \leq k\}$

Then $\min\{k : \mathcal{I}(w) \text{ is } k\text{-generated for all } w \in W\} = \sigma(R) + 1$ *and*

$$\sigma(A_n) = \sigma(B_n) = \sigma(C_n) = \sigma(F_4) = 3,$$
$$\sigma(D_n) = \sigma(E_n) = 4$$

Proof uses reduced expressions... end up reducing to elements like $s_1 s_3 s_2$ in A_3 and $s_1 s_2 s_3 s_0$ in D_4

Stellar root systems arise "naturally"!

Work in progress and open questions

- Is there a combinatorial proof of Billey-Postnikov pattern avoidance criterion for smoothness in types ADE using bound on $\sigma(S)$? (Conjecture: yes)

- What about other properties? For instance, is there a bound on chordality for inversion sets? (Conjecture: no)

Thanks!