Entanglement requirements in non-local games

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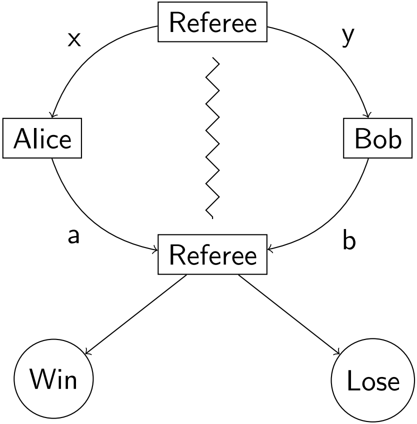
Non-local games (aka Bell-type experiments)



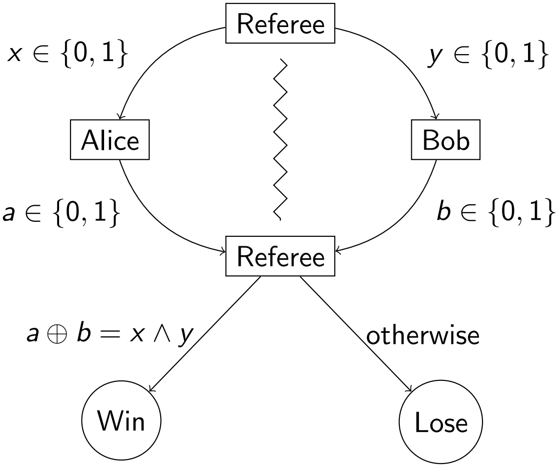
Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

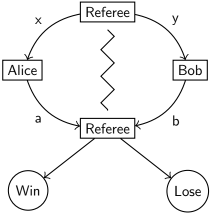
Complication: players cannot communicate while the game is in progress

Example: the CHSH game



Non-local games more formally

A non-local game consists of:



Finite input sets: $\mathcal{I}_X$, $\mathcal{I}_Y$

Finite output sets: $\mathcal{O}_X$, $\mathcal{O}_Y$

A prob. distribution π on $\mathcal{I}_X \times \mathcal{I}_Y$

A function $V : \mathcal{O}_X \times \mathcal{O}_Y \times \mathcal{I}_X \times \mathcal{I}_Y \to \{0, 1\}$

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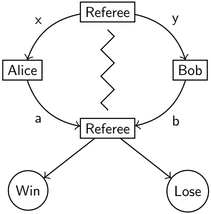

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Interpretation:

If Alice and Bob win on inputs (x, y) and outputs (a, b) then $V(a, b|x, y) = 1$.

Otherwise $V(a, b|x, y) = 0$.

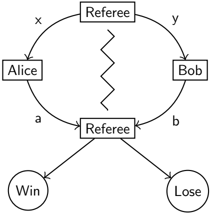
What can the players do?

Alice and Bob can choose their strategy ahead of time.

Deterministic (classical) strategy:

Alice outputs a_x on input x

Bob outputs b_y on input y



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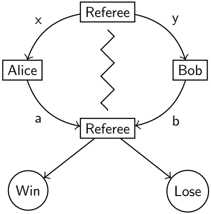

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The winning probability for this strategy S is

$$\omega(S) = \sum_{x\mathcal{I}_A, y\in\mathcal{I}_B} \pi(x,y)V(a_x, b_y|x,y).$$

The *classical value* of the game G is

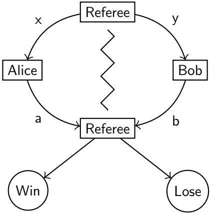
$$\omega^c(G) = \max\{\omega(S) : \text{ deterministic strategies } S\}.$$

What can the players do?

Quantum strategy:

Alice and Bob share quantum state $|\psi\rangle \in H_A \otimes H_B$

Choose outputs according to PVMs $\{P_a^x\}$, $\{Q_b^y\}$



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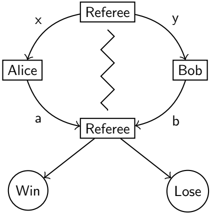

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The quantum value of the game G is

$$\omega^q(G) = \sup\{\omega(S) : \text{ quantum strategies } S\}.$$

Note: no bound on dim H_A, H_B assumed

Why do we care about non-local games?

- Bell inequalities:

  $\omega^c(CHSH) \leq 3/4$, whereas $\omega^q(CHSH) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$.

  Can violate $\omega^c \leq 3/4$ in experiment!

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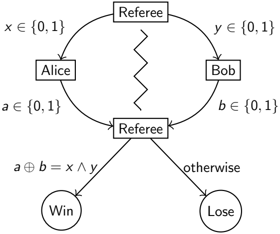
- Non-local games are simple examples of distributed quantum tasks with quantum advantage
- Basis for complexity classes MIP^*, etc.
- Self-testing / device independence:

  For some games G, achieving $\omega^q(G)$ or $\omega^q(G) - \epsilon$ can require states or strategies of a certain form.

  Can certify entanglement, and more complicated things

Self-testing example: CHSH



PVM with two outcomes = ± 1-valued observable (unitary U with $U^2 = 1$)

Strategy for CHSH = state $|\psi\rangle$ and observables A_0, A_1, B_0, B_1 for Alice and Bob respectively

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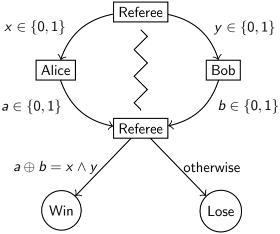

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Strategy for CHSH $=$ state $|\psi\rangle$ and observables A_0, A_1, B_0, B_1 for Alice and Bob respectively

Theorem (Tsirelson)

Strategy $A_0, A_1, B_0, B_1, |\psi\rangle$ is optimal iff $A_0 A_1 = -A_1 A_0$, $B_0 B_1 = -B_1 B_0$, and $|\psi\rangle$ is a Bell state plus ancilla.

Theorem (RUV rigidity lemma)

Any ϵ-optimal strategy is $O(\sqrt{\epsilon})$ close to an optimal strategy.

Fundamental questions

Despite interest in non-local games, there are many basic things about non-local games we don't know:

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Despite interest in non-local games, there are many basic things about non-local games we don't know:

- Can we compute $\omega^q(G)$?
- How much entanglement is required to achieve $\omega^q(G)$ or $\omega^q(G) - \epsilon$?
- Tsirelson problem: can we do better with commuting operator strategies?
- What is the power of MIP^*? Is it decidable?

Focus of this talk:

How much entanglement is required to achieve $\omega^q(G)$ or $\omega^q(G) - \epsilon$?

Bounded entanglement is not enough: there are games with $O(n)$ questions requiring dimension $2^{\Omega(n)}$ to play optimally

Robust version for ϵ-optimal strategies: Ostrev-Vidick

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Is a finite amount of entanglement required for every fixed G?

Finite dimensions are not sufficient for variants of non-local games: [LTW13], [MV13], [RV15]

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Point of this talk: there are non-local games for which no finite-dimensional Hilbert space suffices to achieve $\omega^q(G)$

A (seemingly) simpler question?

Recall that in CHSH, Bob's observables in optimal strategies must satisfy $B_0 B_1 = -B_1 B_0$

Question:

Given a set of algebraic conditions C, is there a non-local game such that, for all optimal strategies, Bob's observables satisfy all conditions in C?

Stronger version:

... such that optimality is equivalent to satisfying conditions in C?

Connection with group theory: linear system games

Start with $m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$

Inputs: Alice receives $1 \leq i \leq m$ (equation)
Bob receives $1 \leq j \leq n$ (variable)

Outputs: Alice: assignment to variables x_k with $A_{ik} \neq 0$
Bob: assignment to variable x_j

Win if Alice's assignment satisfies equation i, and
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Classically: can play perfectly iff $Ax = b$ has a solution

(Play perfectly = win with probability 1)

Quantum: can play perfectly for some $Ax = b$ with no solution

Quantum solutions of $Ax = b$

Observables X_j such that

1. $X_j^2 = I$ for all j
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

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Theorem (Cleve-Mittal)

Let G be the game for linear system $Ax = b$. Then G has a perfect (tensor-product) strategy if and only if $Ax = b$ has a finite-dimensional quantum solution

Note: because there is no bound on dimension, we could have $\omega^q(G) = 1$ without there being a perfect strategy

Connection with group theory

The *solution group* Γ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

1. $X_j^2 = [X_j, J] = J^2 = e$ for all j
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = J^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$

where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity

Theorem (Cleve-Mittal)

Let G be the game for linear system $Ax = b$. Then G has a perfect (tensor-product) strategy if and only if J is non-trivial in some finite-dimensional representation of the solution group Γ.

What about when $\omega^q = 1$?

Theorem (Cleve-Mittal)

Let G be the game for linear system $Ax = b$. Then G has a perfect (tensor-product) strategy if and only if J is non-trivial in some finite-dimensional representation of the solution group Γ.

Is there an algebraic criterion for $\omega^q(G) = 1$?

Partial solution: look at approximate representations of Γ

An ϵ-approximate representation of a finitely-presented group $\langle S : R \rangle$ is a homomorphism $\phi : \mathrm{Free}(S) \to \mathcal{U}(\mathbb{C}^n)$ such that

$$\|\phi(r) - 1\| \leq \epsilon$$

for all $r \in R$.

Characterization of perfect strategies

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Theorem (S)

If J is non-trivial in approximate representations of Γ, then $\omega^q(G) = 1$ (and converse for max-ent. states)

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Idea: find a solution group where J is non-trivial in approximate representations, but trivial in exact representations

Then $\omega^q(G) = 1$, but not achieved on a finite-dimensional space

What groups are solution groups?

There are non-residually finite groups, i.e. groups with elements which are non-trivial but trivial in all finite-dimensional representations

Example (A non-residually finite group)

$K = \langle x, y, a, b : xyx^{-1} = y, yay^{-1} = b, yby^{-1} = a \rangle.$

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Problem: embedding theorem does not necessarily preserve property of being non-trivial in approximate representations

A better embedding theorem

Definition

A linear-plus-conjugacy group is a solution group with added relations of the form $x_i x_j x_i = x_k$.

Conjugacy relations add expressive power:

Example

The symmetric group S_3 $S_3 = \langle a, b : a^2 = b^2 = e, (ab)^3 = e \rangle$ is a (type of) linear-plus-conjugacy group.

To see this, add new generator Z and replace $ababab = e$ with $bab = Z, aZa = b$.

A better embedding theorem

Definition

A linear-plus-conjugacy group is a solution group with added relations of the form $x_i x_j x_i = x_k$.

Theorem

If K is a linear-plus-conjugacy group, then there is an embedding of K in a solution group Γ such that

- *generators of K map to generators of Γ,*
- *J_K maps to J_Γ,*
- *any d-dimensional ϵ-representation ϕ of K maps to an Nd-dimensional $O(\epsilon)$-representation ψ of Γ with $\psi|_K = \phi^{\oplus N}$, for some $N \geq 1$.*

Entanglement requirements for games

Applying embedding theorem to variant of previous example:

Theorem (S)

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The end!