Tsirelson's problem and linear system games

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includes joint work with Richard Cleve and Li Liu

A speculative question

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But... many models in quantum mechanics and quantum field theory require infinite-dimensional Hilbert spaces (e.g. CCR)

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Could nature be "intrinsically" infinite-dimensional?

Answer: Probably not

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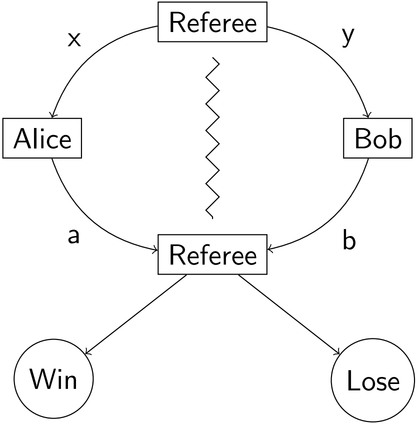
Could nature be "intrinsically" infinite-dimensional?

Answer: Probably not

But if it was... could we recognize that fact in an experiment?

(For instance, in a Bell-type experiment?)

Non-local games (aka Bell-type experiments)



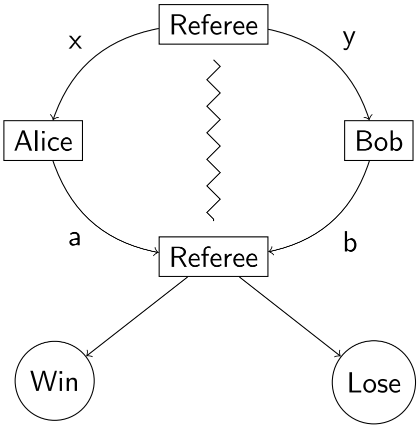
Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Complication: players cannot communicate while the game is in progress

Non-local games ct'd



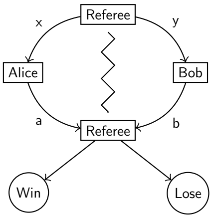
Suppose game is played many times, with inputs drawn from some public distribution π

To outside observer, Alice and Bob's strategy is described by:

$P(a, b|x, y) =$ the probability of output (a, b) on input (x, y)

Correlation matrix: collection of numbers $\{P(a, b|x, y)\}$

What can $P(a, b|x, y)$ be?



$P(a, b|x, y) =$ the probability of output (a, b) on input (x, y)

n questions, m answers: $\{P(a, b|x, y)\} \subset \mathbb{R}^{m^2 n^2}$

<u>Classically</u>

$$P(a, b|x, y) = p_a^x \cdot q_b^y$$

Probability that Alice outputs a on input x

Same for Bob

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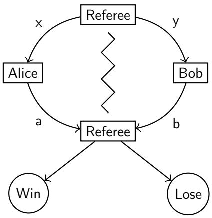

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Classically

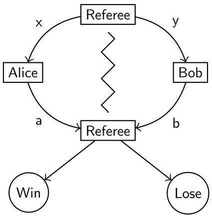
$$P(a, b|x, y) = \sum_i \lambda_i \cdot p_a^{xi} \cdot q_b^{yi}$$

Shared randomness

Probability that Alice outputs a on input x

Same for Bob

What can $P(a, b|x, y)$ be?



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Quantum

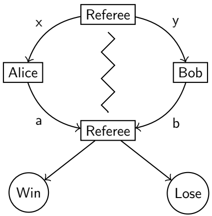
$$P(a, b|x, y) = \langle\psi| \; M_a^x \; \otimes \; N_b^y \; |\psi\rangle$$

Alice's measurement on input x

Bob's measurement on input y

shared state on $H_1 \otimes H_2$

What can $P(a, b|x, y)$ be?



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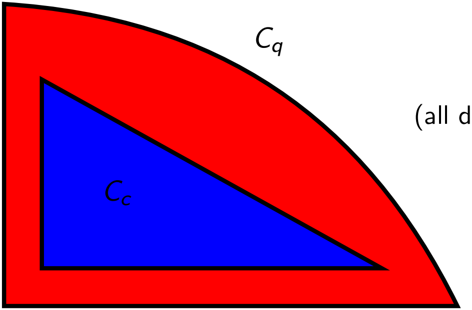
tensor product

Why? axiom of quantum mechanics for composite systems

Bell inequalities



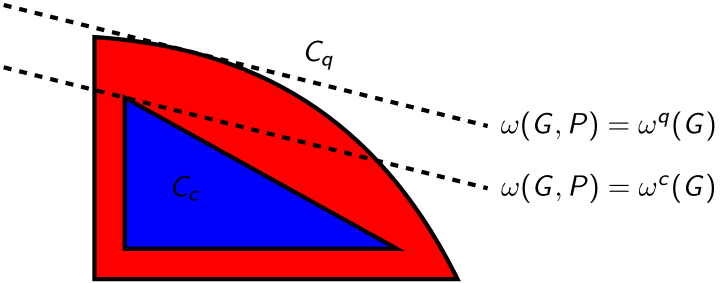
(all diagrams are schematic)

$C_c(m,n) =$ set of classical correlation matrices

$C_q(m,n) =$ set of quantum correlation matrices

Both are convex subsets of $\mathbb{R}^{m^2 n^2}$.

Bell inequalities ct'd

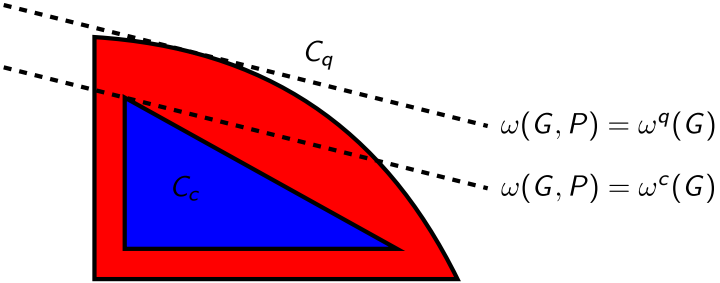


$\omega(G,P) =$ probability of winning game G with correlation P

$\omega^c(G) =$ maximum winning probability for $P \in C_c(m,n)$

$\omega^q(G) =$ same thing but with $C_q(m,n)$

Bell inequalities ct'd



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(1) $C_c \subsetneq C_q$, and

(2) we can (theoretically) show this in an experiment

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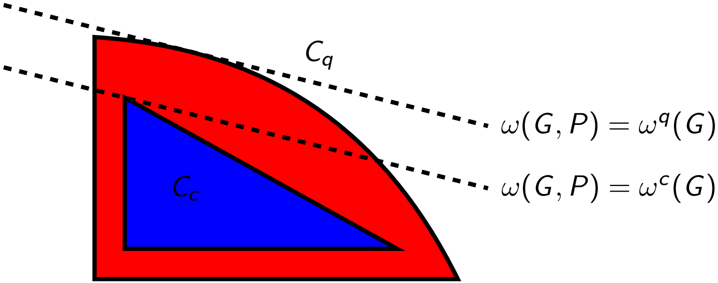

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Bell's theorem + many experiments: this happens!

Finite versus infinite-dimensional

Quantum correlations:

$$P(a, b|x, y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle$$

where $|\psi\rangle \in H_1 \otimes H_2$

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H_1, H_2 must be finite-dimensional

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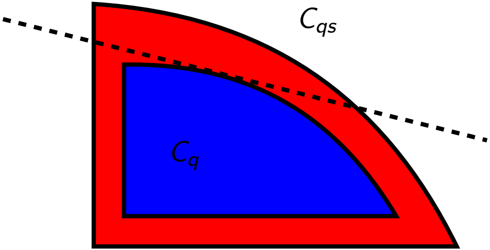
Correlation set C_{qs} :

H_1, H_2 allowed to be infinite-dimensional

(the ‘s’ stands for ‘spatial tensor product’)

Finite versus infinite-dimensional ct'd

Can we separate C_q from C_{qs} with a Bell inequality?



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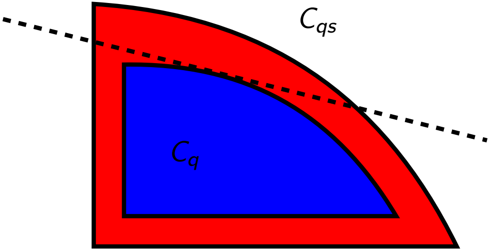
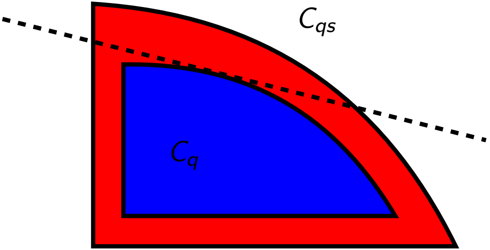


NO!

This is the wrong picture

How is this picture wrong?



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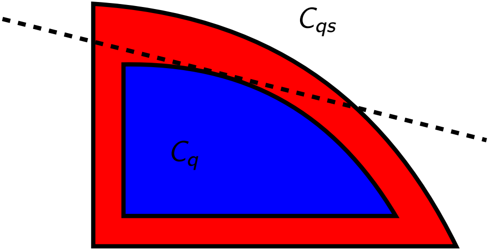

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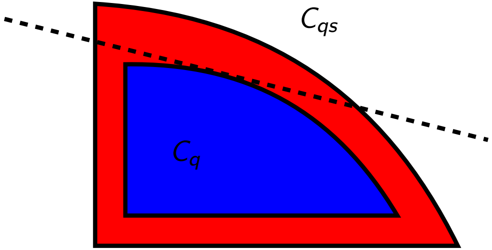


C_q and C_{qs} are not known to be closed.

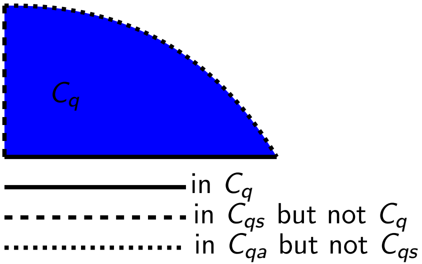
Even worse: $\overline{C_{qs}} = \overline{C_q}$

New correlation set $C_{qa} := \overline{C_q}$

contains limits of finite-dimensional correlations
indistinguishable from C_q and C_{qs} in experiment

The real picture

Could look like:



We know $C_q \subseteq C_{qs} \subseteq C_{qa}$ ... but nothing else!

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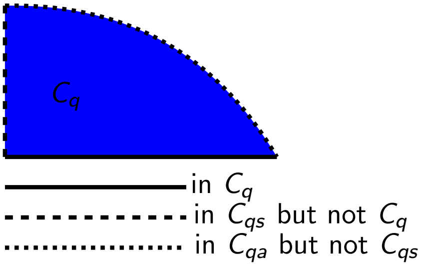

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Fortunately, this is not the end of the story

We've assumed that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$... maybe this is too restrictive

Commuting-operator model

Another model of composite systems

Correlation set C_{qc}:

$$P(a, b|x, y) = \langle\psi| M_a^x \cdot N_b^y |\psi\rangle$$

where

(1) $|\psi\rangle$ belongs to a joint Hilbert space H (possibly infinite-dimensional)

(2) Measurements commute: $M_a^x N_b^y = N_b^y M_a^x$ for all x, y, a, b

'qc' stands for 'quantum-commuting'

What do we know about C_{qc}

Correlation set C_{qc}: $P(a,b|x,y) = \langle\psi| M_a^x \cdot N_b^y |\psi\rangle$

C_{qc} is closed!

Get a hierarchy $C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$ of convex sets

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If H is finite-dimensional, then $\{P(a, b|x, y)\} \in C_q$

$$\text{Can find } H_1, H_2 \text{ such that } H = H_1 \otimes H_2,$$

$$M_a^x \cong \widetilde{M}_a^x \otimes I \text{ and } N_b^y \cong I \otimes \widetilde{N}_b^y \text{ for all } x, y, a, b$$

This argument doesn't work if H is infinite-dimensional

Tsirelson's problem(s)

$$\overbrace{\underbrace{C_q \subseteq C_{qs} \subseteq C_{qa}}_{\text{tensor product}} \underbrace{\subseteq}_{\text{weak}} \underbrace{C_{qc}}_{\text{commuting operator}}}^{\text{strong}}$$

Tsirelson problems: is C_t, $t \in \{q, qs, qa\}$ equal to C_{qc}

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These are fundamental questions

1. Comparing two axiom systems:

   Strong Tsirelson: is $C_q = C_{qc}$?

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These are fundamental questions

1. Comparing two axiom systems:

   Strong Tsirelson: is $C_q = C_{qc}$?

2. Is $\omega^q(G) < \omega^{qc}(G)$ for any game?

   Equivalent to weak Tsirelson: is $C_{qa} = C_{qc}$?

What do we know?

$$\begin{tikzcd}[column sep=small]
C_q \arrow[rrrrrr, bend left, dash, "\text{strong}"] & \subseteq & C_{qs} & \subseteq & C_{qa} \arrow[rr, bend right, dash, "\text{weak}"'] & \subseteq & C_{qc}
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Theorem (Ozawa, JNPPSW, Fr)

$C_{qa} = C_{qc}$ if and only if Connes' embedding problem is true

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Theorem (S)

$C_{qs} \neq C_{qc}$

Other fundamental questions

1. Resource question:

   A non-local game G is a computational task

   Bell's theorem: can do better with entanglement

   Can G be played optimally with finite Hilbert space dimension?

   Yes $\iff$ $C_q = C_{qa}$ (in other words, is C_q closed?)

   Variants of games: finite dimensions do not suffice [LTW13],[MV14],[RV15]

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   Variants of games: finite dimensions do not suffice [LTW13],[MV14],[RV15]

2. Can we compute $\omega^q(G)$ or $\omega^{qc}(G)$?

   (what is the power of MIP^*?)

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General cases of other questions completely open!

Undecidability

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NPA hierarchy: there is no computable function

$$L : \text{ Games } \to \mathbb{N}$$

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NPA hierarchy: there is no computable function

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We still don't know: can we compute $\omega^{qc}(G)$ to within some given error?

(Ji '16: this problem is MIP^*-complete)

If weak Tsirelson is true, then ω^{qc} *is* computable in this stronger sense

Undecidability comes from exact error?

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Consequence of undecidability of $\omega^{qc} < 1$ due to Tobias Fritz:

> quantum logic (first order theory for projections on Hilbert spaces) is undecidable

Quantum logic is undecidable

Theorem (Tobias Fritz)

The following problem is undecidable:
Given $n \geq 1$ and a collection of subsets $\mathcal{C}$ of $\{1, \ldots, n\}$, determine if there are self-adjoint projections $P_1, \ldots, P_n$ such that

$$\sum_{i \in S} P_i = I, \quad P_i P_j = P_j P_i = 0 \text{ if } i \neq j \in S$$

for all $S \in \mathcal{C}$.

Proof: follows from undecidability of $\omega^{qc} < 1$

Builds on Acín-Fritz-Leverrier-Sainz '15.

Two theorems

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$C_{qs} \neq C_{qc}$

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But: proof follows from a single theorem in group theory

Connection with group theory comes from linear system games

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Such games go back to Mermin-Peres magic square, more recently studied by Cleve-Mittal, Ji, Arkhipov

Quantum solutions of $Ax = b$

Observables X_j such that

1. $X_j^2 = I$ for all j
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

(We've written linear equations multiplicatively)

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Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if $Ax = b$ has a finite-dimensional quantum solution*
- *G has a perfect strategy in C_{qc} if and only if $Ax = b$ has a quantum solution*

Group theory ct'd

The *solution group* Γ of $Ax = b$ is the group generated by $X_1, \ldots, X_n, J$ such that

1. $X_j^2 = [X_j, J] = J^2 = e$ for all j
2. $\prod_{j=1}^n X_j^{A_{ij}} = J^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$

where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- *G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Groups and local compatibility

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For instance, S_3 is generated by a, b subject to the relations

$$a^2 = b^2 = e, (ab)^3 = e$$

If $ab = ba$, then $(ab)^3 = a^3 b^3 = ab$

So relations imply $a = b$, and S_3 becomes $\mathbb{Z}_2$

Group embedding theorem

Solution groups satisfy local compatibility

Nonetheless:

Solution groups are as complicated as general groups

Theorem (S)

Let G be any finitely-presented group, and suppose we are given J_0 in the center of G such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where Γ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$.

How do we prove the embedding theorem?

Linear system $Ax = b$ over $\mathbb{Z}_2$ equivalent to labelled hypergraph:

Edges are variables

Vertices are equations

v is adjacent to e if and only if $A_{ve} \neq 0$

v is labelled by $b_i \in \mathbb{Z}_2$

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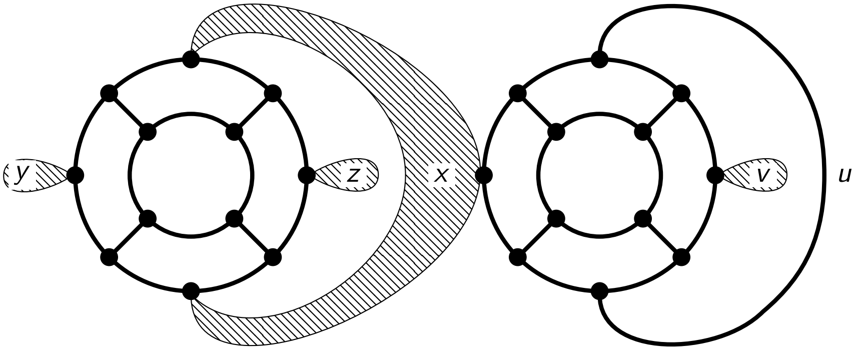
Given finitely-presented group G, we get Γ from a linear system

But what linear system?

Can answer this pictorially by writing down a hypergraph?

The hypergraph by example



$$\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle$$

Further directions

1. Further refinements to address C_q vs C_{qa}
2. Is $\omega^q(G) < 1$ decidable?

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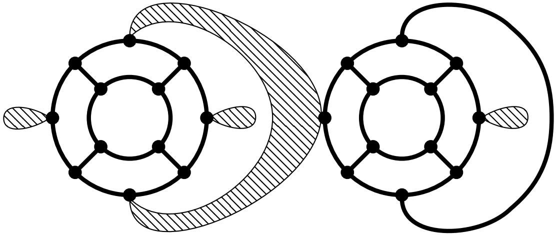
2. Is $\omega^q(G) < 1$ decidable?
3. Embedding theorem: for any f.p. group G, get a non-local game such that Alice and Bob are forced to use G to play perfectly

   (Caveat: but might need to use infinite-dimensional commuting-operator strategy to achieve this)

   Applications to self-testing / device independent protocols?

The end



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Thank-you!

Extra slide: Higman's group

$$G = \langle a, b, c, d : aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2$$

Only finite-dimensional representation is the trivial representation

On the other hand, a, b, c, d are all non-trivial in G