Isomorphism type of Schubert varieties

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Generalized Cartan matrices

A GCM is an $n \times n$ matrix A such that

- $A_{ii} = 2$ for all $i = 1, \ldots, n$,
- $A_{ij} \leq 0$ if $i \neq j$, and
- if $A_{ij} = 0$ then $A_{ji} = 0$.

Examples:

$$\text{type A}: \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -3 & -7 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

Weyl groups

Starting with $n \times n$ GCM A, the Weyl group $W(A)$ is the group

$$\langle s_1, \ldots, s_n : s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \text{ for } 1 \leq i \neq j \leq n \rangle$$

with

$$m_{ij} = \begin{cases} 2 & A_{ij}A_{ji} = 0 \\ 3 & A_{ij}A_{ji} = 1 \\ 4 & A_{ij}A_{ji} = 2 \\ 6 & A_{ij}A_{ji} = 3 \\ \infty & A_{ij}A_{ji} \geq 4 \end{cases}$$

Example: $W(A_n) = S_{n+1}$, the permutation group

Flag varieties

From a GCM A, can also construct:

- A Kac-Moody group $\mathcal{G} = \mathcal{G}(A)$, including Cartan and Borel subgroups $T \subseteq \mathcal{B}$.

  Example: $\mathcal{G}(A_n) = \mathrm{GL}_n \mathbb{C}$, $T =$ diagonal invertible matrices, $\mathcal{B} =$ upper triangular invertible matrices.

- The full flag variety $\mathcal{X}(A) = \mathcal{G}/\mathcal{B}$.

  For A_n, get the space

  $$Fl(n) = \{0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n \subsetneq E_{n+1} = \mathbb{C}^{n+1}\}.$$

  In general, $\mathcal{X}(A)$ can be infinite-dimensional.

Schubert varieties

From a GCM A, can also construct:

- A Kac-Moody group $\mathcal{G} = \mathcal{G}(A)$, including Cartan and Borel subgroups $T \subseteq \mathcal{B}$.
- The full flag variety $\mathcal{X}(A)$ of A, defined by $\mathcal{G}/\mathcal{B}$.
  In general, $\mathcal{X}(A)$ can be infinite-dimensional.
- Schubert varieties $\mathcal{X}(w; A)$ indexed by $w \in W(A)$. These are finite-dimensional normal projective T-varieties stratifying $\mathcal{X}(A)$.

$\mathcal{X}(w; A_n)$ is the closure of $\mathcal{B}F_w$, where $F_w = (E_0, \ldots, E_n)$ is defined by $E_i = span\{e_{w(1)}, \ldots, e_{w(i)}\}$.

Natural question: When are the Schubert varieties $\mathcal{X}(w; A)$ and $\mathcal{X}(w'; A')$ isomorphic as algebraic varieties?

Motivation: are there smooth varieties in affine type $\widetilde{A}_n$ that do not appear in finite type?

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Possible answer: diagram isomorphism

(Example: $\mathcal{X}(s_i) \cong \mathbb{P}^1$)

When are $\mathcal{X}(w; A)$ and $\mathcal{X}(w'; A')$ isomorphic as algebraic varieties?

$s_{i_1} \cdots s_{i_k}$ is a reduced word if there no way to write w as a product of fewer simple reflections s_i

$S(w) = \{1 \leq i \leq n : s_i \text{ appears in reduced word for } w\}$

Suppose $w \in W(A)$, $w' \in W(A')$, and there is a bijection $\sigma : S(w) \to S(w')$ such that

- $A_{st} = A'_{\sigma(s)\sigma(t)}$ for all $s, t \in S(w)$
- the iso $W(A)_{S(w)} \to W(A')_{S(w')}$ sends $w \mapsto w'$.

Then $\mathcal{X}(w; A) \cong \mathcal{X}(w'; A')$.

Example: In A_3, $\mathcal{X}(s_1 s_2 s_1) \cong \mathcal{X}(s_2 s_3 s_2)$.

Are diagram isomorphisms the only isomorphisms?

Look at $X(s_1 s_2; A)$ with $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$

$$\begin{array}{ccc} \mathbb{P}^1 \cong \mathcal{X}(s_2) & \longrightarrow & \mathcal{X}(s_1 s_2) \\ & & \downarrow \\ & & \mathcal{X}(s_2) \cong \mathbb{P}^1 \end{array}$$

$\mathcal{X}(s_1 s_2)$ is a Hirzebruch surface Σ_n

($\Sigma_n \cong \Sigma_m$ if and only if $m = n$)

Multiplication table on H^2:

	ξ^{s_1}	ξ^{s_2}
ξ^{s_1}	0	$\xi^{s_1 s_2}$
ξ^{s_2}	$\xi^{s_1 s_2}$	$a\xi^{s_1 s_2}$

$\mathcal{X}(s_1 s_2)$ is Σ_a.

b is irrelevant

Conclusion: no!

When are $\mathcal{X}(w; A)$ and $\mathcal{X}(w'; A')$ isomorphic as algebraic varieties?

Theorem (Richmond-S)

The following are equivalent:

1. $\mathcal{X}(w; A) \cong \mathcal{X}(w'; A')$
2. *there is an isomorphism* $H^*(\mathcal{X}(w; A)) \to H^*(\mathcal{X}(w'; A'))$ *which preserves the Schubert basis*
3. *there is a bijection* $\sigma : S(w) \to S(w')$ *and a reduced expression* $w = s_{i_1} \cdots s_{i_k}$ *such that*
    - $s'_{\sigma(i_1)} \cdots s'_{\sigma(i_k)}$ *is a reduced expression for* w', *and*
    - $A_{i_j i_{j'}} = A_{\sigma(i_j)\sigma(i_{j'})}$ *for all* $j < j'$

Hard direction: (3) implies (1)

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Why? No T-variety structure

Proof: (1) implies (2)?

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$H^*(\mathcal{X}(w; A)$ spanned by Schubert classes ξ^v, $v \leq w$ in Bruhat order

These classes are the extremal rays of the effective cone of $\mathcal{X}(w; A)$

Proof: (2) implies (3)?

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Proof: (2) implies (3)?

Given an algebraic variety X which is promised to be a Schubert variety, can we construct A and $w \in W(A)$ such that $X \cong \mathcal{X}(w; A)$?

Answer: yes, identify extremal rays of effective cone in $H^*(X)$ with Schubert classes and recover w from rules for Schubert calculus

$H^2(X)$ spanned by ξ^{s_i}, $i \in S(w) \implies$ can identity $S(w)$

Can recover Bruhat order and right descents from Chevalley-Monk formula for $\xi^{s_i}\xi^v \implies$ can get reduced expression

A_{ij} shows up in structure constants if and only if $s_i s_j \leq w$ in Bruhat order

Further questions

Does the same answer apply to parabolic Schubert varieties $\mathcal{X}^J(w; A)$?

Difficulty: $H^2(\mathcal{X}^J(w; A))$ no longer spanned by ξ^{s_i}, $i \in S(w)$

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The end!