Lecture 1: Algebras

William Slofstra

September 11, 2019

1 Introduction

In this lecture, we'll cover some of the basic ideas behind $*$-algebras and their representations. Our approach to approximate representation theory will be fairly concrete, and we really won't use much more than linear algebra at most points. But it's helpful to have the abstact structures in mind when working with approximate representations.

2 Algebras

2.1 Basic definitions

Let $\mathbb{F}$ be a field (typically $\mathbb{R}$ or $\mathbb{C}$). An $\mathbb{F}$-algebra is an $\mathbb{F}$-vector space $\mathcal{A}$ with a multiplication operation, which is just a linear map $m : \mathcal{A} \otimes_{\mathcal{F}} \mathcal{A} \to \mathcal{A}$. We usually write ab or $a \cdot b$ for $m(a \otimes b)$.

Note that if $\mathcal{A}$ is an algebra, then

$$(a+b) \cdot c = m((a+b) \otimes c) = m(a \otimes c + b \otimes c) = ac + bc$$

for all $a, b, c \in \mathcal{A}$, and similarly $a(b+c) = ab + ac$. Also, if $c \in \mathbb{F}$, then

$$(ca) \cdot b = m(ca \otimes b) = m(a \otimes cb) = a \cdot (cb).$$

So using the tensor product in the definition of multiplication requires multiplication to be distributive, and scalar multiplication to commute with the product.

Exercise 2.1. *Show that there is a linear isomorphism between linear maps $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, and bilinear maps $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, sending $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ to $\tilde{m} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, where $\tilde{m}(a, b) = m(a \otimes b)$.*

Examples of algebras include:

(1) The space $M_n\mathbb{F}$ of $n \times n$ matrices over $\mathbb{F}$, with the usual matrix multiplication.

(2) the space $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H.

(3) The space of $n \times n$ matrices $M_n\mathbb{F}$ with multiplication defined by $m(a,b) = [a,b] := ab - ba$. This algebra is an example of a *Lie algebra*.

(4) Polynomials $\mathbb{F}[x_1, \ldots, x_n]$ in n variables over $\mathbb{F}$.

(5) Non-commutative polynomials $\mathbb{F}\langle x_1, \ldots, x_n \rangle$ in n variables over $\mathbb{F}$.

(6) Let X be a set, and let $\mathbb{F}X$ be the free vector space over X. By definition, X is a basis for $\mathbb{F}x$, and hence the set $\{a \otimes b : (a,b) \in X \times X\}$ is a basis for $\mathbb{F}X \otimes \mathbb{F}X$. So any function $m : X \times X \to X$ extends linearly to a function $\mathbb{F}X \otimes \mathbb{F}X \to \mathbb{F}X$ via the rule

$$\sum_{(a,b)\in X\times X} c_{ab} a \otimes b \mapsto \sum_{(a,b)\in X\times X} c_{ab} m(a,b),$$

and hence defines an algebra on $\mathbb{F}X$.

Example (6) can be used to give a rigorous definition of non-commutative polynomials. Let X be the space of finite sequences $(i_1, \ldots, i_k)$, where $k \geq 0$ and $i_j \in \{1, \ldots, n\}$ for all j. Let m be the concatenation function $X \times X \to X$, so

$$m((i_1, \ldots, i_k), (j_1, \ldots, j_l)) = (i_1, \ldots, i_k, j_1, \ldots, j_l).$$

If we write sequences $(i_1, \ldots, i_k)$ as $x_{i_1} \cdots x_{i_k}$ and the empty sequence as 1, then we see that $\mathbb{F}X$ formalizes the notion of non-commutative polynomials. By replacing $\{1, \ldots, n\}$ with an arbitrary set S, we can use this construction to define non-commutative polynomials $\mathbb{F}\langle S \rangle$ over any set of variables S, finite or infinite. Replacing X with the set of multisets gives a construction of the usual (commutative) polynomial algebra.

As the above examples show, there is no requirement that algebras be finite-dimensional. A word of caution for those used to working with Hilbert spaces: just because a space is infinite-dimensional doesn't mean we start taking infinite sums. Every Hilbert space H has a Hilbert basis B, and the elements of the space can be written as possibly-infinite sums $\sum_{x\in B} c_x x$ where the family $(c_x)_{x\in\mathcal{B}}$ is square-summable. In contrast, with an algebraic (sometimes called

Hamel) basis, we only take finite sums. For instance, the free vector space $\mathbb{F}X$ is defined to be the space of formal linear combinations

$$\sum_{x \in X} c_x x$$

where all but finitely many elements of the family $(c_x)_{x \in X}$ are zero. Spaces like $\mathbb{F}X$ are not Hilbert spaces, and thus don't necessarily have a Hilbert basis. (There is a Hilbert-space analog of $\mathbb{F}X$ called the free Hilbert space over X, in which X is a Hilbert basis.)

An algebra $\mathcal{A}$ is *associative* if $a(bc) = (ab)c$ for all $a, b, c \in \mathcal{A}$, and *unital* if there is an element $1 \in \mathcal{A}$ (called a *unit*) such that $1a = a1 = a$ for all $a \in \mathcal{A}$. A unit is unique if it exists, since if 1 and $1'$ are both units, then

$$1 = 11' = 1'.$$

If $1 \neq 0$ then $\mathcal{A}$ contains a copy of $\mathbb{F}$. If $1 = 0$, then $\mathcal{A}$ must be the zero algebra, which is the space 0 with the only possible multiplication, namely $0 \cdot 0 = 0$.

Examples (1), (2), (4), and (5) above are both associative and unital, while (3) is neither. The algebras in example (6) are not necessarily associative or unital in general. We'll use the term *algebra* to refer to unital associative algebras by default.

2.2 Quotients

A *(two-sided) ideal* in an algebra $\mathcal{A}$ is a subspace $\mathcal{I}$ such that $a\mathcal{I} \subset \mathcal{I}$ and $\mathcal{I}a \subset \mathcal{I}$ for all $a \in \mathcal{A}$. Given an ideal $\mathcal{I}$, the quotient space $\mathcal{A}/\mathcal{I} = \{a + \mathcal{I} : a \in \mathcal{A}\}$ is an algebra with multiplication $(a + \mathcal{I})(b + \mathcal{I}) = ab + \mathcal{I}$. The quotient space $\mathcal{A}/\mathcal{I}$ is zero if and only if $\mathcal{I} = \mathcal{A}$. If $\mathcal{A}$ is unital, then this occurs if and only if $1 \in \mathcal{I}$. When context is clear, we usually just write a for the element $a + \mathcal{I}$ of a quotient $\mathcal{A}/\mathcal{I}$. Another common notational convention is to write $a + \mathcal{I}$ as $\overline{a}$.

Given a subset R of an algebra $\mathcal{A}$, the ideal $\langle R \rangle$ generated by R is the smallest ideal containing R, or equivalently the subspace

$$\{\sum_{i=1}^{k} a_i r_i b_i : k \geq 0, a_i, b_i \in \mathcal{A}, r_i \in R \text{ for all } 1 \leq i \leq k\}.$$

(This expression for $\langle R \rangle$ assumes that $\mathcal{A}$ is unital. For non-unital algebras it must be modified a bit.)

Starting from an algebra $\mathcal{A}$, we can construct new algebras by picking any subset R, and taking the quotient $\mathcal{A}/\langle R\rangle$. This quotient is non-zero if and only if $1 \notin \langle R\rangle$, meaning that $1 \neq \sum_{i=1}^{k} a_i r_i b_i$ for any $k \geq 0$, $a_i, b_i \in \mathcal{A}$, $r_i \in R$, $1 \leq i \leq k$. We are particularly interested in taking quotients of non-commutative polynomial algebras. Given a set of variables S and a subset $R \subset \mathbb{F}\langle S\rangle$, we let

$$\mathbb{F}\langle S : R\rangle := \mathbb{F}\langle S\rangle/\langle R\rangle.$$

If S and R are finite sets, then $\mathbb{F}\langle S : R\rangle$ is called a *finitely presented algebra*. The sets S and R are part of the data of a finitely presented algebra. In fact, whenever we talk about a *presented* algebra, we mean an algebra $\mathbb{F}\langle S : R\rangle$ with some specified sets S and R.

Example 2.2. *Another possible way to define the polynomial algebra* $\mathbb{F}[x_1, \ldots, x_n]$ *is as the quotient*

$$\mathbb{F}\langle x_1, \ldots, x_n : x_i x_j = x_j x_i \text{ for } 1 \leq i, j \leq n\rangle.$$

The relation $x_i x_j = x_j x_i$ *means we quotient by* $x_i x_j - x_j x_i$. *In general, we can write* $L = R$ *for the polynomial* $L - R$.

Example 2.3. *One of the enjoyable aspects of finitely presented algebras is that we can write down any generators and relations we want. For instance, we can define an algebra*

$$\mathbb{F}\langle x, y : xy - yx = 1\rangle$$

even though there are no matrices x, y *that satisfy the equation* $xy - yx = 1$ *(see homework below).*

Example 2.4. *By writing down arbitrary relations, we might end up with the zero algebra. For a trivial example,*

$$\mathbb{F}\langle x, y : x - 1, x - y, y\rangle = 0$$

because $1 = -(x - 1) + (x - y) + y$ *is in the defining ideal.*

2.3 Homomorphisms

A *homomorphism* between algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that $\phi(ab) = \phi(a)\phi(b)$ and $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. A homomorphism is an *isomorphism* if it is bijective, and we say that $\mathcal{A}$ is isomorphic to $\mathcal{B}$ (in notation, $\mathcal{A} \cong \mathcal{B}$) if there is an isomorphism $\phi : \mathcal{A} \to \mathcal{B}$.

Exercise 2.5. *Show that* $\mathcal{A} \cong \mathcal{B}$ *if and only if there are homomorphisms* $\phi : \mathcal{A} \to \mathcal{B}$ *and* $\psi : \mathcal{B} \to \mathcal{A}$ *such that* $\psi \circ \phi = Id_{\mathcal{A}}$ *and* $\phi \circ \psi = Id_{\mathcal{B}}$*.*

Example 2.6. *If* $\mathcal{I}$ *is an ideal, then the quotient map* $q : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ *is a homomorphism, with the property that* $q(x) = 0$ *if and only if* $x \in \mathcal{I}$*.*

Example 2.7. *Let* $p \in \mathbb{F}\langle x_1, \ldots, x_n \rangle$*, and let* $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$*, where* $\mathcal{A}$ *is an algebra. If*

$$p = \sum_{\tilde{i}} c_{\tilde{i}} x_{i_1} \cdots x_{i_k},$$

then the evaluation $p(a)$ *of* p *at* a *is*

$$\sum_{\tilde{i}} c_{\tilde{i}} a_{i_1} \cdots a_{i_k}$$

(just like evaluation of commutative polynomials, we plug a_i *in for* x_i*).*

Fixing $a \in \mathcal{A}^n$*, the map*

$$\mathbb{F}\langle x_1, \ldots, x_n \rangle \to \mathcal{A} : p \mapsto p(a)$$

is a homomorphism (called evaluation at a*), and in fact is the unique homomorphism* $\mathbb{F}\langle x_1, \ldots, x_n \rangle$ *sending* $x_i \mapsto a_i$*.*

Conversely, if $\phi : \mathbb{F}\langle x_1, \ldots, x_n \rangle \to \mathcal{A}$ *is a homomorphism, then* ϕ *must be evaluation at* a *where* $a_i = \phi(x_i)$ *for* $i = 1, \ldots, n$*. As a result, there is a bijection between homomorphisms* $\mathbb{F}\langle x_1, \ldots, x_n \rangle \to \mathcal{A}$ *and the set of* n*-tuples* $\mathcal{A}^n$*.*

For a set of variables S*, we can say that homomorphisms* $\mathbb{F}\langle S \rangle \to \mathcal{A}$ *correspond to functions* $\mathcal{A}^S$*.*

If $\mathcal{I}$ is an ideal of $\mathcal{A}$, and $\phi : \mathcal{A}/I \to \mathcal{B}$ is a homomorphism, then we can compose ϕ with the quotient map $q : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ to get a homomorphism $\phi \circ q : \mathcal{A} \to \mathcal{B}$ with the property that $\phi \circ q(r) = \phi(0) = 0$.

Exercise 2.8. *Let* $R \subset \mathcal{A}$*, let* $q : \mathcal{A} \to \mathcal{A}/\langle R \rangle$ *be the quotient homomorphism, and let* $\tilde{\phi} : \mathcal{A} \to \mathcal{B}$ *be a homomorphism. Show that there is a homomorphism* $\phi : \mathcal{A}/\langle R \rangle \to \mathcal{B}$ *with* $\tilde{\phi} = \phi \circ q$ *if and only if* $\tilde{\phi}(r) = 0$ *for all* $r \in R$*, and that the homomorphism* ϕ *is unique if it exists.*

In particular, the exercise tells us that homomorphisms $\mathbb{F}\langle S : R \rangle \to \mathcal{A}$ correspond to homomorphisms $\phi : \mathbb{F}\langle S \rangle \to \mathcal{A}$ with the property that $\phi(r) = 0$ for all $r \in R$. By the example above, every such homomorphism must be an evaluation map. If ϕ is evaluation at $a \in \mathcal{A}^S$, then $\phi(r) = r(a)$, so we get the following proposition:

Proposition 2.9. *There is a bijection between homomorphisms* $\mathbb{F}\langle S : R\rangle \to \mathcal{A}$ *and elements* $a \in \mathcal{A}^S$ *such that* $r(a) = 0$ *for every* $r \in R$.

A *representation* of an algebra $\mathcal{A}$ on a Hilbert space H is a homomorphism $\mathcal{A} \to \mathcal{B}(H)$.

Homework 1. *Show that the algebra* $\mathbb{C}\langle x, y : xy - yx = 1\rangle$ *does not have a representation on a Hilbert space.*

An algebra is *finitely presentable* if it is isomorphic to a finitely presented algebra. Alternatively, we can say that $\mathcal{A}$ is finitely presentable if there is an isomorphism $\mathbb{F}\langle S : R\rangle \to \mathcal{A}$, where S and R are finite sets. This means that there must be elements $a_s \in \mathcal{A}$ for each $s \in S$, such that $r(a) = 0$ for all $r \in R$, every element of $\mathcal{A}$ can be written as a polynomial in the elements a_s, and such that the relations R express the only relations between different expressions for any given element as a polynomial in the a_s's.

When we say that an algebra is finitely presentable, we mean that it has a finite presentation, but we don't remember the algebra as part of the presentation. For instance,

$$\mathbb{C}\langle x, y : y = 0\rangle \text{ and } \mathbb{C}\langle x\rangle$$

are two different presentations for the noncommutative polynomial algebra in one variable. However, as finitely present*ed* algebras, these two algebras are different.

3 $*$-algebras

4 Normed algebras