Tsirelson's problem and linear system games

William Slofstra

IQC, University of Waterloo

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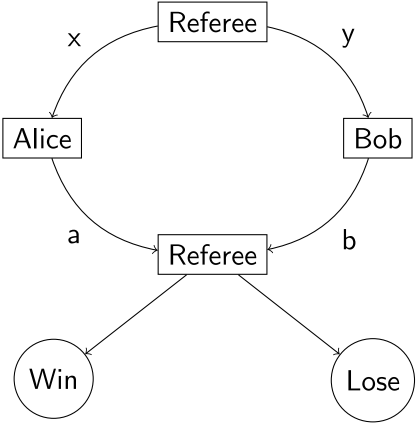
includes joint work with Richard Cleve and Li Liu

Non-local games

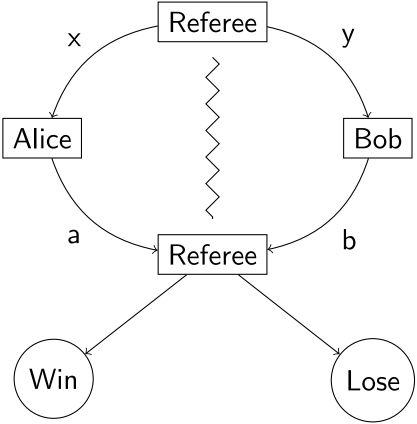


Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Non-local games



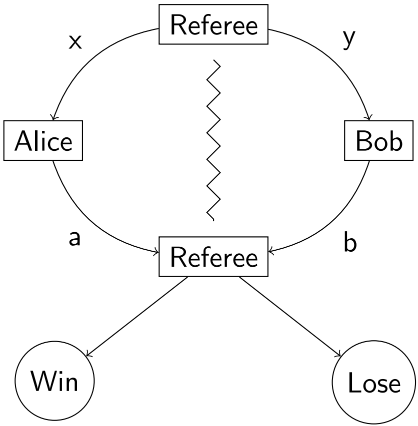
Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Complication: players cannot communicate while the game is in progress

Strategies for non-local games



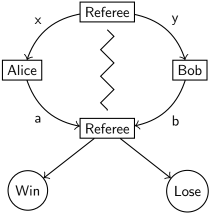
Suppose game is played many times, with inputs drawn from some public distribution π

To outside observer, Alice and Bob's strategy is described by:

$P(a, b|x, y)$ = the probability of output (a, b) on input (x, y)

Correlation matrix: collection of numbers $\{P(a, b|x, y)\}$

Classical and quantum strategies



$P(a, b|x, y)$ = the probability of output (a, b) on input (x, y)

Value of game ω = winning probability using strategy $\{P(a, b|x, y)\}$

What type of strategies might Alice and Bob use?

Classical: can use randomness, flip coin to determine output.

Correlation matrix will be $P(a, b|x, y) = A(a|x)B(b|y)$.

Quantum: Alice and Bob can share entangled quantum state

Bell's theorem: Alice and Bob can do better with an entangled quantum state than they can do classically

Quantum strategies

How do we describe a quantum strategy?

Use axioms of quantum mechanics:

- Physical system described by (finite-dimensional) Hilbert space
- No communication $\Rightarrow$ Alice and Bob each have their own (finite dimensional) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$
- Hilbert space for composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Shared quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$
- Alice's output on input x is modelled by measurement operators $\{M_a^x\}_a$ on $\mathcal{H}_A$
- Similarly Bob has measurement operators $\{N_b^y\}_b$ on $\mathcal{H}_B$

Quantum correlation: $P(a, b|x, y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle$

Quantum correlations

Set of quantum correlations:

$$C_q = \Big\{ \{P(a,b|x,y)\} : P(a,b|x,y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle \text{ where}$$
$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \text{ where } \mathcal{H}_A, \mathcal{H}_B \text{ fin dim'l}$$
$$M_a^x \text{ and } N_b^y \text{ are projections on } \mathcal{H}_A \text{ and } \mathcal{H}_B$$
$$\sum_a M_a^x = I \text{ and } \sum_b N_b^y = I \text{ for all } x, y \Big\}$$

Two variants:

1. C_{qs}: Allow $\mathcal{H}_A$ and $\mathcal{H}_B$ to be infinite-dimensional
2. $C_{qa} = \overline{C_q}$: limits of finite-dimensional strategies

Relations: $C_q \subseteq C_{qs} \subseteq C_{qa}$

Commuting-operator model

Another model for composite systems: *commuting-operator model*

In this model:

- Alice and Bob each have an algebra of observables $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A}$ and $\mathcal{B}$ act on the joint Hilbert space $\mathcal{H}$
- $\mathcal{A}$ and $\mathcal{B}$ commute: if $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $ab = ba$.

This model is used in quantum field theory

Correlation set:

$$C_{qc} := \Big\{ \{P(a,b|x,y)\} : P(a,b|x,y) = \langle\psi| M_a^x N_b^y |\psi\rangle , \\ M_a^x N_b^y = N_b^y M_a^x \Big\}$$

Hierarchy: $C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$

Tsirelson's problem

strong

$$C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$$

weak

Two models of QM: tensor product and commuting-operator

Tsirelson problems: is C_t, $t \in \{q, qs, qa\}$ equal to C_{qc}

Fundamental questions:

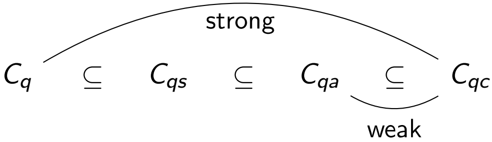
1. What is the power of these models?

   Strong Tsirelson: is $C_q = C_{qc}$?

2. Are there observable differences between these two models, accounting for noise and experimental error?

   Weak Tsirelson: is $C_{qa} = C_{qc}$?

What do we know?



Theorem (Ozawa, JNPPSW, Fr)

$C_{qa} = C_{qc}$ *if and only if Connes' embedding problem is true*

Theorem (S)

$C_{qs} \neq C_{qc}$

Other fundamental questions

Question: Given a non-local game, can we compute the optimal value ω_t over strategies in C_t, $t \in \{qa, qc\}$?

Theorem (Navascués, Pironio, Acín)

Given a non-local game, there is a hierarchy of SDPs which converge in value to ω_{qc}

Problem: no way to tell how close we are to the correct answer

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

Two theorems

Theorem (S)

$C_{qs} \neq C_{qc}$

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

Proofs: make connection to group theory via linear system games

Linear system games

Start with $m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$

$\implies$ Get a non-local game G, and

$\implies$ a *solution group* Γ

Γ: Group generated by $X_1, \ldots, X_n$, satisfying relations

1. $X_j^2 = [X_j, J] = J^2 = e$ for all j
2. $\prod_{j=1}^{n} X_j^{A_{ij}} = J^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$.

Quantum solutions of $Ax = b$

Solution group Γ: Group generated by $X_1, \ldots, X_n$, satisfying relations

1. $X_j^2 = [X_j, J] = J^2 = e$ for all j
2. $\prod_{j=1}^n X_j^{A_{ij}} = J^{b_i}$ for all i
3. If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$.

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- *G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Group embedding theorem

Theorem (Cleve-Mittal,Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- *G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- *G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Theorem (S)

Let G be any finitely-presented group, and suppose we are given J_0 in the center of G such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where Γ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$.

How do we prove the embedding theorem?

Theorem (S)

Let G be any finitely-presented group, and suppose we are given J_0 in the center of G such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where Γ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$.

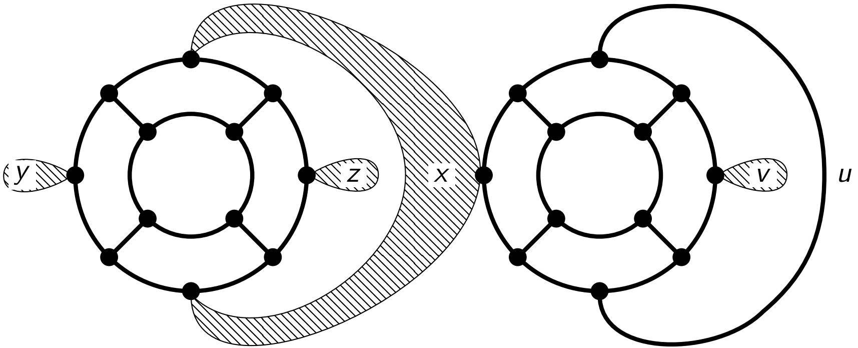
Given finitely-presented group G, we get Γ from a linear system

But what linear system?

Linear systems over $\mathbb{Z}_2$ correspond to vertex-labelled hypergraphs

So we can answer this pictorially by writing down a hypergraph...

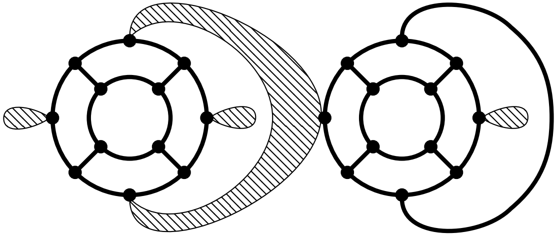
The hypergraph by example



$$\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \cdots = v^2 \rangle$$

does not include preprocessing

The end



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Thank-you!