Entanglement in non-local games

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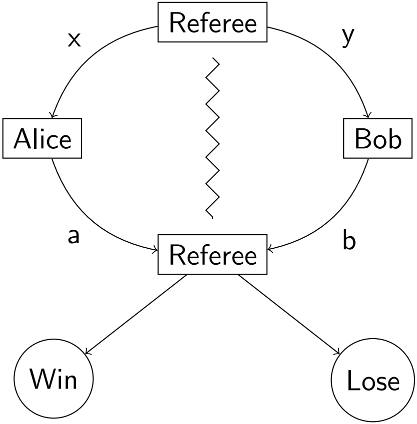
Non-local games (aka Bell scenarios)



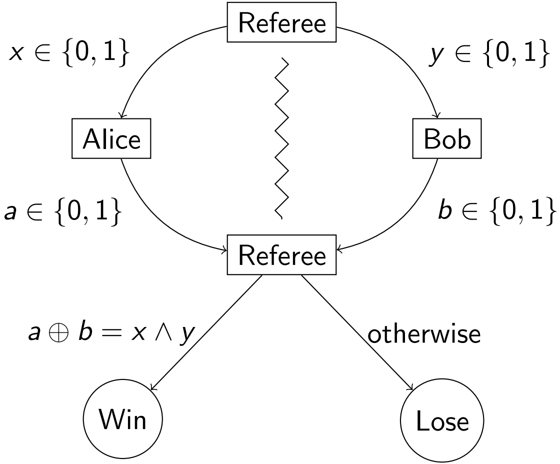
Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Complication: players cannot communicate while the game is in progress

Example: the CHSH game



Compare with:

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$$

Best classical strategy has winning probability $3/4$

Best entangled strategy has winning probability ≈ 0.85.

Example: Mermin-Peres magic square game

x_1	x_2	x_3	0
x_4	x_5	x_6	0
x_7	x_8	x_9	0
1	1	1	

Alice receives either a row or column
Returns binary assignment to variables in that row or column

Bob receives a variable x_i, $1 \leq i \leq 9$
Returns a binary assignment to that variable

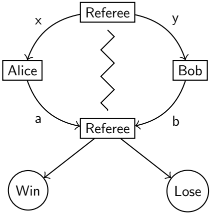
Players win if Alice's output sums to either 0 (row) or 1 (column), and Alice and Bob's output is consistent.

Best classical strategy has winning probability 26/27

Best entangled strategy has winning probability 1

Non-local games more formally

A non-local game $\mathcal{G}$ consists of:



Finite input sets: $\mathcal{I}_X$, $\mathcal{I}_Y$

Finite output sets: $\mathcal{O}_X$, $\mathcal{O}_Y$

A prob. distribution π on $\mathcal{I}_X \times \mathcal{I}_Y$

A function $V : \mathcal{O}_X \times \mathcal{O}_Y \times \mathcal{I}_X \times \mathcal{I}_Y \to \{0, 1\}$
(1 =win, 0=lose)

$\omega^c(\mathcal{G})$:= the optimal winning probability with a classical strategy

$\omega^q(\mathcal{G})$:= the optimal winning probability with a quantum strategy

If $\omega^c(\mathcal{G}) < \omega^q(\mathcal{G})$, then can think of $\mathcal{G}$ as a distributed computational task with quantum advantage

Entanglement requirements

We'd like a resource theory for non-local games

How much "entanglement" $E(\mathcal{G}, \epsilon)$ is required to attain $\omega^q(\mathcal{G}) - \epsilon$?

Possible resources: **local Hilbert space dimension (Schmidt rank)**, von Neumann entropy, "non-locality"

Examples: $E(CHSH, 0) = 2$, $E(MSQ, 0) = 4$.

Both games are rigid, meaning that there is $\epsilon_0 > 0$ such that $E(\mathcal{G}, \epsilon) = E(\mathcal{G}, \epsilon_0)$ for all $\epsilon < \epsilon_0$.

Can we find a game $\mathcal{G}$ and an $\epsilon \geq 0$ such that $E(\mathcal{G}, \epsilon) \geq d$?

How many questions n or answers m does $\mathcal{G}$ need to have to get dimension d?

Some examples (not a complete list):

- Brunner et. al., 2008: original question
- Junge-Palazuelos, 2011: $m = n$ to get $d = \sqrt{n}/\log(n)$ with multiplicative gap $O(d)$
- Ostrev-Vidick, 2016: $m = 2$, $\epsilon = O(1/n^{5/2})$ to get $d = 2^{\Omega(\sqrt{n})}$
- Natarajan-Vidick, 2017: $m =$constant, $\epsilon =$constant, $d = \Omega(n)$

Key idea of Ostrev-Vidick: Game for which near-optimal strategies can be turned into approximate representations of Clifford algebra

$$\mathbb{C}\langle X_1, \ldots, X_n : X_i^2 = 1, X_i X_j = -X_j X_i, i \neq j\rangle.$$

Approximate representations have dimension $2^{\Omega(n)}$.

More recently, we have started to find games $\mathcal{G}$ where $E(\mathcal{G}, \epsilon) \to +\infty$ as $\epsilon \to 0$.

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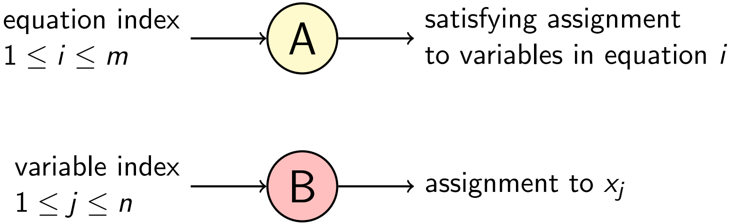
- S-Vidick (2017): two-player game $\mathcal{G}$ such that $\Omega(1/\epsilon^{1/6}) \leq E(\mathcal{G}, \epsilon) \leq O(1/\epsilon^{1/2})$.
- Ji-Leung-Vidick (2018): three-player game $\mathcal{G}$ such that $2^{\Omega(1/\epsilon^c)} \leq E(\mathcal{G}, \epsilon) \leq 2^{O(1/\epsilon)}$.
- S (2018): two-player game $\mathcal{G}$ such that Hilbert space dimension required to get $\omega^q(\mathcal{G}) \geq 1 - \epsilon$ is $2^{\Omega(1/\epsilon^c)}$.
- Fitzsimons-Ji-Vidick-Yuen (2018): 15-player game $\mathcal{G}$ such that $E(\mathcal{G}, \epsilon) \geq 2^{O(1/\epsilon^c)}$.

Linear system non-local games

$m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$
$\implies$ two-player non-local game $\mathcal{G}$ (Aravind, Cleve-Mittal)



Inputs chosen at random

Players win if Alice's output is consistent with Bob's output

Ex: Magic square is a linear system non-local game

Quantum solutions of linear systems

Non-local game $\mathcal{G}$ for $Ax = b$ has perfect classical strategy if and only if $Ax = b$ has a solution

Theorem (Cleve-Mittal, Cleve-Liu-S)

$\mathcal{G}$ has a perfect quantum strategy
if and only if $Ax = b$ has a finite-dimensional quantum solution

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Quantum solution:

Collection of unitaries $X_1, \ldots, X_n \in \mathcal{U}(\mathcal{H})$ such that

1. $X_j^2 = 1$ for all j,
2. $\prod_{j=1}^n X_j^{A_{ij}} = (-1)^{b_i}$ for all $i = 1, \ldots, n$,
3. $X_j X_k = X_k X_j$ if $A_{ij}, A_{ik} \neq 0$ for some i.

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if and only if $Ax = b$ has a finite-dimensional quantum solution
if and only if $J \neq 1$ in the group $\Gamma(A, b)$

Solution group of $Ax = b$

$$\Gamma(A,b) = \left\langle x_1, \ldots, x_n, J \; : \; \begin{array}{l} x_j^2 = 1 = [x_j, J] = J^2 \text{ for all } j \\ \prod_j x_j^{A_{ij}} = J^{b_i}, i = 1, \ldots, m \\ [x_j, x_k] = 1 \text{ if } A_{ij}, A_{ik} \neq 0, \text{ some } i \end{array} \right\rangle$$

Approximate representations

An ϵ-approximate representation of a finitely-presented group $\langle S : R \rangle$ is a homomorphism $\phi : \text{Free}(S) \to \mathcal{U}(\mathbb{C}^n)$ such that

$$\|\phi(r) - 1\|_f \leq \epsilon$$

for all $r \in R$.

Theorem (S-Vidick, Ozawa)

Let Γ be a solution group of a linear system game $\mathcal{G}$.

d-dimensional ϵ-representations of Γ with $J \mapsto -1$ $\longleftrightarrow$ *$O(\text{poly}(\epsilon))$-perfect strategies for $\mathcal{G}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$*

Hyperlinear profile $\text{hlp}(w, \delta, \epsilon)$: smallest dimension d such that there is a d-dimensional ϵ-representation ϕ with $\|\phi(w) - 1\|_f \geq \delta$.
Conclusion: $E(\mathcal{G}, \epsilon) \simeq \text{hlp}(J, 2, \epsilon^c)$

Constructing interesting solution groups

How to find solution groups Γ where $\text{hlp}(J, 2, \epsilon)$ grows fast?

Theorem (S)

Every finitely-presented group embeds in a solution group.

If $x \in G \subseteq H$, then $\text{hlp}_G(x, \delta, \epsilon) \leq \text{hlp}_H(x, \delta, \epsilon)$
So for lower bounds on $E(\mathcal{G}, \epsilon)$, we just need to find groups with fast-growing hyperlinear profile

- S-Vidick: there is a group G and $w \in G$ with $\text{hlp}(w, 2, \epsilon) \geq 1/\epsilon^{2/3}$.
- S: there is a group G and $w \in G$ with $\text{hlp}(w, 2, \epsilon) \geq 2^{\Omega(1/\epsilon^c)}$. Proof uses stability of approximate representations of Clifford algebra plus quantitative version of Higman's embedding theorem due to Birget, Ol'shanskii, Rips, Sapir

Upper bounds

Big question: is $E(\mathcal{G}, \epsilon)$ always finite for $\epsilon > 0$?
Equivalent to Connes embedding problem (Fritz, JNPPSW, Ozawa)
If so, $E(\mathcal{G}, \epsilon)$ has a computable upper bound (and so does MIP^*).

- Ji-Leung-Vidick (2018): three-player game G such that $2^{\Theta(1/\epsilon^c})) \leq E(G, \epsilon) \leq 2^{O(1/\epsilon)}$.
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The end!